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ABSTRACT

This publication contains reference materials and activities for teachers in a staff development program in New York City, teaching grades 3 and 6. The program combines content and methodology. Sections in the publication include: (1) Student Verbalization Resources; (2) Manipulative Resources; (3) Probability, Statistics, and Graph Resources; (4) Resources for Instruction; (5) Estimation Resources; (6) Research Resources; (7) Mathematical Concept Development; and (8) Sample Mathematics Masters. Different sections contain different kinds of materials including background and evaluation materials, activities, lists, duplication masters, and teaching tips. (RH)

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Grades 3 and 6 Mathematics

Volume Two

Staff Development Program
1986-1987

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Grades 3 and 6 Mathematics

Volume Two

Staff Development Program
1986-1987

New York City Board of Education • Division of Curriculum and Instruction • Office of Curriculum Development and Support

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INTRODUCTION

The Grades 3 and 6 Mathematics Staff Development Program for 1986/87 is a combination of content and methodology. It is important to have a knowledge of what to teach as well as how to teach. Teachers will focus their attention on content, lesson planning, questioning, homework and testing through the use of manipulatives. Teachers must become proficient in these essential areas in order to deliver effective instruction.

As participants master these significant aspects of teaching, a central thread runs through these areas. Student verbalization allows students to become active participants in the process of teaching and learning. Research has shown that expressing one's thinking through verbalization brings thinking to the conscious level. Students who take a verbally active part in each lesson through discussing, questioning, explaining and justifying their solutions have allowed their thoughts to be formulated and polished through the additional sense of hearing. Students who are asked to verbalize what strategies they have used to solve a set of problems are more successful on a subsequent set of related problems than students who have not been asked to put their strategies into words.

It is essential that teachers totally involve students in all aspects of the learning process. The teacher must have students respond during class in full sentence answers so that they are reinforcing not only what is done to solve a problem but also when and how the method is to be applied. The teacher must have students come to the board, write on the board and verbalize the strategies for solutions to the homework, class problems and teacher-made tests. Mathematics instruction becomes most effective when students actively discuss what they are learning.

Through verbalization, students see mathematics as a close knit system of understandable ideas, principles and processes. Students are more likely to retain mathematical learning over time if meaningful discussion activities comprise more than half of their mathematics instruction. Teachers must aim for a regular and consistent involvement of students through verbalization in order to assure meaningful instruction in mathematics. The crucial relationship between student verbalization and understanding is essential to insure student retention.

ACKNOWLEDGMENTS

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Student Verbalization Resources

Language and Vocabulary *

Mathematics Spoken Here: A Case for Language and Vocabulary Instruction in Mathematics

by Dr. Lelon R. Capps and
Dr. Martha S. Gage

Mathematics Spoken Here
Se Habla Matemáticas
数学是我们这里的语言。

There is mounting evidence that learning and understanding mathematical language and vocabulary plays a vital role in mastering mathematics. Students who understand mathematical terms and who can verbalize problems tend to have a better grasp of concepts and be more successful in both computation and problem solving. How refreshing it would be if, just as travelers in foreign countries find "English Spoken Here," "Spanish Spoken Here,"—we could all post a similar sign in our math classrooms, "Math Spoken Here."

Aiken (1972) believes that "a significant degree of overlap between verbal and mathematical abilities remains unexplained" and that the crucial variable may not be general intelligence but rather linguistic ability. Kane (1974) suggests that in order for students to become competent readers of mathematics, they need special skills developed through instruction in mathematical language. Heilman, Blair and Rupley (1981) point out that when students can recognize words instantly they can focus on understanding the material instead of having to stop to analyze words (and symbols). The question is what is the exact role of language and vocabulary in the learning of mathematics?

Johnson (1944) studied the effect of learning mathematical vocabulary on the solution of problems in which the vocabulary appeared. This carefully designed study found that learning the vocabulary allowed the experimental group to outper-

4 from "Current Issues in Mathematics" Houghton Mifflin Company, Lawrenceville, New Jersey 08648.

form the control group to a significant degree. The conclusion was not surprising. Mathematics vocabulary, studied in context, has a profound effect on performance.

When we look closely at mathematical problems the importance of vocabulary is obvious. Consider the following statement that might appear on a standardized test:

Find the perimeter of the triangle.

One must know the precise meaning of the mathematical words *perimeter* and *triangle* to choose a correct answer. Lyda and Duncan (1967) proved that direct study of *quantitative* vocabulary contributes significantly to better mathematical word problem-solving ability. These researchers recommended that all teachers incorporate direct instruction of quantitative vocabulary into the mathematics class period.

Let us consider the implications of this in the classroom. A good reading teacher always introduces new vocabulary before teaching a new story. The same should happen in mathematics. Several minutes should be devoted to the new vocabulary that occurs in the mathematics lesson. Sight recognition cards could be made for terms and symbols. New terms should be used naturally and repeatedly in the lesson context so they become familiar tools to the child. And finally, textbooks in mathematics must provide consistent and frequent attention to the vocabulary and language skills needed for proper mathematics learning.

It is interesting to note that teachers, when teaching reading, verbalize each new vocabulary word as well as use that word in a sentence. This is just what is needed in the mathematics lesson! *Verbalization* is equally important in mathematics. Consider the following example:

When children complete this example without verbalizing they count the 3 circles and write 3. Then they count the 2 triangles and write 2. Finally they count the five shapes and write 5.

If the children never read the completed sentence, and do not hear "Three plus two equals five," they may not make any connection between the three and two and the resulting five. They also miss the chance to relate the printed symbols to spoken words.

Asking children to formulate a word problem from the circle and triangle art will help them gain familiarity with "word problems" long before they are asked to read and solve them. When children verbalize the problem

There are three circles. There are two triangles. How many shapes are there together?

they gain invaluable preparation for translating word sentences to number sentences—a key skill for solving word problems.

Now let us consider *meaning in context*. The teacher of reading must take elaborate precautions with words such as *tear, polish* and *read*. Special attention is given to using these words in sentences.

I have a *tear* in my eye. There is a *tear* in my glove.
Polish your shoes. *Polish* food is tasty.
Read your lesson. Have you *read* your lesson?

Obviously the child must learn to decipher word meaning from the context of the sentence.

The same is true for the symbol 2. Precautions and efforts must be made to ensure that the child can distinguish its meaning in contexts like these:

10^2 ½ 10_2 246 cm^2

In 10^2 the ^{2} can be read "squared," "second power," "exponent 2" and "10 taken as a factor 2 times." In ½ the 2 reads as "half." Small wonder that children have difficulty.

Children need to be prepared to deal with many words semantically (meaning) and phonetically (orally) in order to grasp and deal with arithmetic problems and discussions. Many words are not found in their reading programs. Stauffer (1966) found in his investigation 1331 words, appearing in the arithmetic vocabulary list made for grades 1, 2 and 3, which did not appear anywhere in the reading vocabulary list made for the same grades. Could there be any more convincing evidence of the need to teach language and vocabulary skills during the mathematics period?

Children learn much of their reading and speaking vocabularies in everyday social interaction. This is not true for mathematics vocabulary. To a major extent, mathematics language can be learned only in the classroom. We must teach it there.

continued...

Language and Vocabulary *continued...*

Most educators are in agreement. Students need instruction in math language and vocabulary to be successful in computation and problem solving. Dahmus (1970) states that teachers and texts must develop students' abilities to translate situations into proper mathematical symbols so the students will be able to read and write mathematics.

Wickelgren (1974) concludes that problems often rely heavily on verbal language and the first and most basic ability in problem solving is to be able to represent or translate the verbal information into symbolic or diagrammatic form.

The moral is clear. Every classroom should be able to post the sign:

"MATHEMATICS SPOKEN HERE." ✖

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Dr. Martha S. Gage *is Chairman of the Department of Education at Ottawa University, Ottawa, Kansas.*

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Mathematics Vocabulary Lists

MATHEMATICS VOCABULARY-PRIMARY

A
add
addend
addition
alike
all
amount
angle
Arabic numerals
area
array
associative property
average

B
base
base ten
between
both

C
calendar
cent
center
centimeter
change
circle
circumference
column
combine
common factor
common multiple
commutative property
compare
compass
composite number
compute
cone
congruent
connect
contain
corner
cost
count
counting numbers
cube
curve
cylinder

D
decimal
degree
denominator
diagonal
diameter
difference
digit
dime
distance
distributive property
divide
dividend
divisor
dollar
double
dozen

E
each
element
empty set
end point
equal

equation
equivalent
estimate
even number
expanded numeral

F
fact
factor
fewer
figure
foot, feet
fraction

G
gallon
geometry
gram
graph
greater than
grid
group

H
half
height
hexagon
horizontal
hour

I
identify
inch
inequality
infinite set
inside
instead
intersect

J
join

K
kilogram
kilometer

L
least
length
less
line
liter

M
many
match
mathematics
measure
median
member
metric
middle
mile
minuend
minus
minute
missing
mixed
model
money
more
multiple
multiplication

N
name
natural order
negative
nickel
number
number line
numerator

O
object
octagon
odd
one-to-one
open
operation
opposite
order
ordinal
ounce

P
pair
parallel
parallelogram
parenthesis, parentheses
pattern
penny
pentagon
percent
perimeter
perpendicular
place holder
place value
plane
plus
point
polygon
positive
pound
prime number
principle
problem
product
property
probability
protractor
pyramid

Q

quadrilateral
quart
quotient

R

radius
range
ratio
ray
reciprocal
rectangle
regroup
related facts
remainder
rename
rhombus
right angle
Roman numeral
round
row

S

same
score
second
segment
sequence
sequence
shaded
shape
short form
side
sign
similar
simple
single
size
solution
solve
some
space
sphere
square
standard numeral
straight
subset
subtract
subtrahend
sum
symbol
symmetrical

T

table
temperature
thermometer
time
times
ton
total
triangle

U

unequal
union
unit
unknown
unnamed

V

value
vertex
vertical
volume

W

week
weight
whole number
width

Y

yard

Z

zero

MATHEMATICS VOCABULARY-INTERMEDIATE

A

abscissa
absolute value
acre
actual
acute angle
acute triangle
additive
additive inverse
adjacent
alternate
altitude
approximately
arc
arithmetic mean
avoirdupois
axis

B

baker's dozen
bar graph
base five
base two
basic facts
billion
binary operation
bisect
broken-line graph

C

calculate
capacity
cardinal
caret
cast out
Celsius
centigrade
centigram
centiliter
central angle
century
circle graph
circumscribe
clock arithmetic
closed curve
closure
commission
common denominator
comparison
complex
computation
concentric
consecutive
constant
construct
convert
coordinate axis
corresponding
cross-product
cross-section
cubed
cubic
currency

D

data
decade
decimal point
decrease
deposit
depth
derive
determine
diagram
dimension
discount
denominator
diagonal
diameter
divisibility
division
dot graph
dry measure
duplicate

E

equality
equiangular triangle
equilateral triangle
error of measurement
estimation
exact
expanded
experiment
exponent
exponential notation
express
extend
exterior

F

face
face value
factor tree
factorial
Fahrenheit
family of facts
finite

fixed
fluid ounce
foci
foot-pound
formula
frequency
frequency table

G
generalization
given
graduated scale
graph of the equation
greatest common factor
greatest possible multiple
gross
gross weight

H
hemisphere
histogram

I
identity property
imply
improper fraction
include
inclusion
increase
inference
input
integer
interest rate
interior
interpret
intersection
interval
inverse
isosceles triangle

K
kiloliter

L
lateral
least common denominator
least common factor
least common multiple
leg of a triangle
linear
line graph
line of symmetry
line segment
lowest terms

M
magic square
markup
matching
maximum
mean
meter stick
micron
midway
minimum
mixed numeral
mode
month
multiplicand
multiplier

N
natural number
negative number
notation
numeral
numeration
numeration system

O
obtuse angle
odd number
open equation
open figure
open sentence
ordered pairs
origin
outcome
output

P
partial
per
perfect number
period
pi
pictograph
plane figure
plot
power
precision
predict
prime factor
procedure
profit
progession
proper fraction
proportion
Pythagorean theorem

Q
quadrant
quantity

R
radian
radii
ranking
rate
rational number
reduce
reflexive property
regular
repeating decimal
restriction
reverse
right triangle

S
satisfies
scale drawing
scalene triangle
scientific notation
sector
semicircle
short division
simplest terms
simplify
skew lines
skip counting
solid
squared
square feet
square inches
square meter
square mile
square root
standard
statute mile
story problem
straight angle
straight edge
subscript
substitute
subtraction
successive
super set
supplementary angle
surface area
system

T
tablespoon
tally
tangent
teaspoon
terminate
terms
theorem
transitive property
trapezoid
trial
triple
two-dimensional

U
undivided
union of sets
universal set
unlimited
unmatched
upper limit

V
variable
vertical angle
vertical axis

W
word problem

X
x-axis

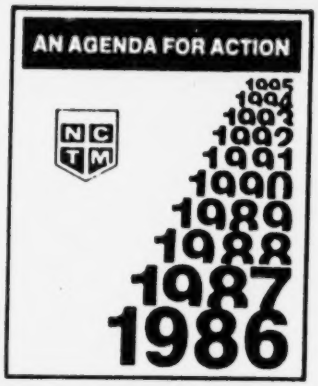
Y
yardstick
y-axis
yearly

Manipulative Resources

One Point of View

Using Manipulatives

By George W. Bright



Manipulatives hold the promise for helping many students understand mathematics, but as with any "cure" they also hold the potential for harm if they are used badly. Careless use of manipulatives may convince students that two mathematical worlds exist, symbols and manipulatives, and that it is acceptable for these worlds to have different rules. An acceptance of this inconsistency is certainly harmful to the development of proper mathematical understanding.

Students may appear to misunderstand concepts when they get different answers with symbols than they do with manipulatives. This discrepancy may, however, be an artifact of the physical constraints of manipulatives. Frequently, manipulatives almost force students to produce the correct answer, so laying the blame on the symbolic world may be inappropriate. Rather, it should be recognized that proper mathematical understanding simply does not exist when students do not see the need for accurate connection between the two worlds.

Indeed, the main emphasis that needs to be made in teaching with manipulatives is that symbols and manipulatives must always reflect the same concepts. Students don't automatically grasp this connection.

The acceptance of this need for consistency seems to have been an important step in my own appreciation for, and use of, mathematics. I suspect the same is true for many people. Many teachers, however, only *implicitly* embed this need for consistency between the two worlds in their teaching.

Oral language may provide one bridge between the two worlds. I believe that many mathematical concepts are developed through abstraction and generalization across multiple instances. However, abstraction need not always be done in a written symbolic mode. It is equally important to abstract in an oral language mode. Manipulatives can provide the experiential base for oral language while simultaneously complementing the symbolic side of mathematics. The experience of touching and moving objects may also serve to relate concepts sequentially, thus implicitly illustrating relationships among concepts. Unless the connection between the two settings is carefully articulated, however, counterproductive tension may be created between them.

Students' misunderstanding of the connection, as with any other misunderstanding, requires that remediation be undertaken. Unfortunately, little seems to be known about how to provide this remediation. But we must not let our lack of knowledge become a justification for substituting comfortable symbol-pushing techniques for instruction with manipulatives. Rather, we should endeavor to become more sensitive to finding adequate ways of incorporating manipulatives into instruction and helping students make appropriate connections between manipulatives and symbols.

Certainly manipulatives should not be used for teaching every concept. Careful use of manipulatives provides opportunities for generalizing across multiple instances of concepts. But as with any attempt to foster transfer of understanding, these general learning skills must be explicitly highlighted in instruction if they are to be learned. Learning how to connect the various parts of mathematics, thereby developing adequate generalization skills, is far more important than learning the particular concepts and skills presented in lessons built around manipulatives. All of us who teach mathematics must learn how to teach these general skills, and one of the tools for doing so is the use of manipulatives. ■

George Bright teaches computer education courses at the University of Houston, Houston, TX 77004, though he remains active in research on ways of improving mathematics instruction.

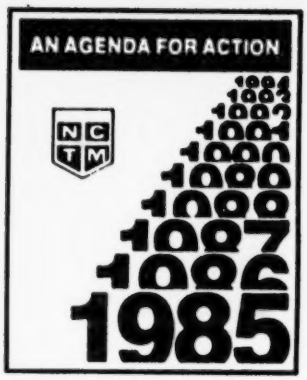
Readers are encouraged to react to these editorials by writing to the author, with copies to the Arithmetic Teacher *for consideration in "Readers' Dialogue." Please double-space all letters that are to be considered for publication.*

This Month's *MT*

Readers of the *Arithmetic Teacher* would enjoy the following articles in this month's *Mathematics Teacher:*

- "The Impact of the Black Death (1348–1405) on World Population: Then and Now," Michael G. Gonzales and William J. Carr
- "Activities: Calculators, Quotients, and Number Patterns," David R. Duncan and Bonnie H. Litwiller
- "Sharing Teaching Ideas: Graphs: Patterns, Periodic Functions, and Problem Solving," Ernest Woodward

One Point of View



Manipulatives Are Good Mathematics!

By **Elizabeth Herbert**
Houston Independent School District, Houston, TX 77027

When asked, "Do you like mathematics? Why or why not?" at the beginning of each school year, a startling 80 to 85 percent of my middle school students would reply that they did not like mathematics because it was boring and because they did not understand it very well. Their negative attitudes appeared to stem from a lack of understanding based on plain boredom with uninteresting classes.

For fifteen years, this informal survey yielded the same answers on the first or second day of school. By week three, the same students no longer dreaded mathematics. To their surprise and my delight, they were enjoying mathematics. Why? Manipulatives.

Manipulatives can and should be used on a regular basis in the mathematics classroom, along with a variety of other tools and methods. On the basis of my experience as a teacher, mathematics department head, instructional supervisor, and workshop leader, I strongly advocate the use of manipulatives for three reasons: manipulatives motivate students; manipulatives stimulate students to think mathematically; and manipulatives informally introduce "big" ideas in mathematics.

Teachers often claim that not enough time is available to use manipulatives, that their use is the same as playing games, and that they are difficult to manage with large numbers of students.

Yes; using manipulatives takes time. Isn't it better to take time at the beginning of a unit to motivate students and spark their thinking than to take time during a unit to bore them—and yourself—with endless drills and worksheets? Thinking time is time well spent.

Yes, using manipulatives feels like playing games. Teachers and students having fun thinking and discovering together is what makes teaching so rewarding. We must be ready to share ourselves, our thought processes, our own mistakes, and our sense of fun with students.

No, manipulatives are not difficult to manage with large numbers of students. First, explain to the entire class the purpose of the manipulative. Show them how it works and then pose a problem or situation. Divide the class into small groups and supervise them as you would any group activity. Circulate among them, help any who really need assistance, but try to let them struggle and experiment. Don't solve problems for them. If any students start flipping rubber bands or throwing rods, isolate them and make them work out of the textbook.

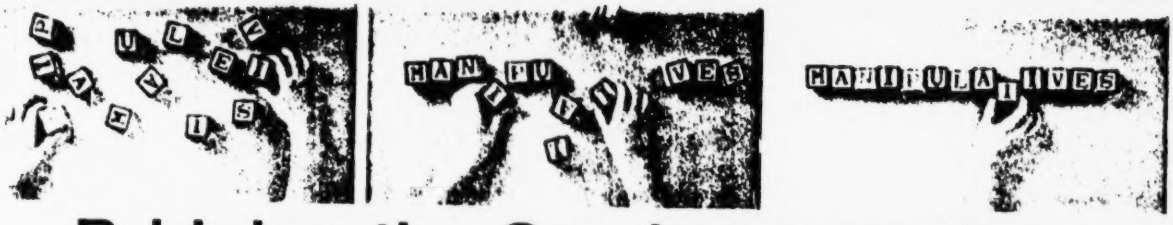
Conquer the fear of losing control of the class. A little noise can be expected when students work together. A few "ahs," "wows," and "I understands" will be music to your ears. Invite the principal to observe. If you're lucky, the principal will join a group and try to help solve a problem.

A favorite definition of mathematics states, "Mathematics is discovering relationships and expressing those relationships in symbolic form." What better way to introduce young children to mathematical thinking than to have them use Cuisenaire rods or fraction bars to discover the algorithms for adding and subtracting fractions? Middle school students can use geoboards to discover and write the formulas for finding the areas of polygons. Algebra students can feel challenged and interested when they are confronted with this proposal: "Let's find a way to determine the distance between these two points on your geoboard." The students use the Pythagorean theorem and then discover, write, and use the distance formula.

In each example, the manipulative provides the situation from which real mathematical thinking arises. The students gain the understanding that may have come using traditional methods after hours of drill and practice. Effective thinking in mathematics depends on understanding, not on remembering words and phrases from a text.

A confining, narrow textbook approach to teaching mathematics is inadequate and inappropriate in today's schools. It does motivate students—to take as few mathematics courses as possible—and often does not help them to become independent, confident problem solvers.

Textbooks transmit known mathematics; they tend to simplify learning by emphasizing the mastery of small, isolated steps. Mathematical thinking is not done in small, isolated steps using finished definitions from a text. Rather, it starts with situations that draw mathematical responses from the students. Manipulatives allow teachers to create those situations and involve the students actively in the creation of mathematics, resulting in improvements in motivation, involvement, understanding, and achievement—overwhelming reasons to believe that manipulatives are good mathematics. ●



Bridging the Gap between the Concrete and the Abstract

By **James W. Heddens**

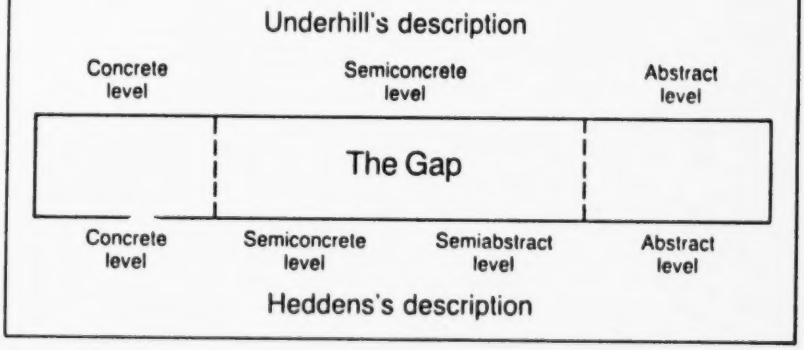
How can teachers help students make the transition from the use of manipulative materials to abstract mathematical symbols? Many students have difficulty understanding mathematics because they are unable to make a connection between the physical world and the abstract world.

Defining the gap

Underhill (1972) defines the stage between the concrete and abstract levels as the semiconcrete stage. Heddens (1984) divides this stage into two levels—semiconcrete and semiabstract. The semiconcrete level is a representation of a real situation; pictures of the real items are used rather than the items themselves. The semiabstract level involves a symbolic representation of concrete items, but the symbols or pictures do not look like the objects for which they stand. Tallies might be used to represent the idea of automobiles, for example.

The gap between concrete and abstract functioning should be considered as a continuum. Helping children bridge this gap is crucial because many children cannot cross it without the teacher's assistance. Figure 1 represents a continuum of learning, with the gap appearing between the concrete and abstract levels. Learners must internalize new knowledge at the concrete level and systematically progress along the continuum to arrive at the abstract representation of that knowledge.

Fig. 1 A continuum of learning from concrete to abstract



Two processes of interaction between reality and the mind are accommodation and assimilation. Piaget refers to these processes as *equilibration*. New knowledge can be assimilated very rapidly by some children. Other children, who must accommodate, or reorganize, their mental structures to incorporate new knowledge, need considerably more time. It takes time to think! While students who must accommodate are processing new material, teachers very often continue to present more new material, thus leaving the student in the lurch. A provision must be made for students to bridge the gap at different rates.

Bridging the gap with questions

Teachers can also help students bridge the gap by very carefully asking questions that will help children move from the concrete to the abstract level. Too many questions in mathematics classes require specific answers and encourage students to remember only facts. The students' thought processes are often ignored when questions are asked. Teachers should continually evaluate the quality and level of the questions that students are asked. Are the questions intellectually stimulating? Do they require students to synthesize before they respond? Questions that match all levels of Bloom's taxonomy (Bloom et al. 1977) should be used. More emphasis should be placed on the "how" and "why" questions and less emphasis on the "what" questions. Through systematic questioning by the teacher, children can begin to develop their own thought processes and use these techniques in their own thinking. Simply using manipulatives in teaching mathematics is not sufficient; teachers must guide children to develop skills in thinking.

Children do not learn mathematics

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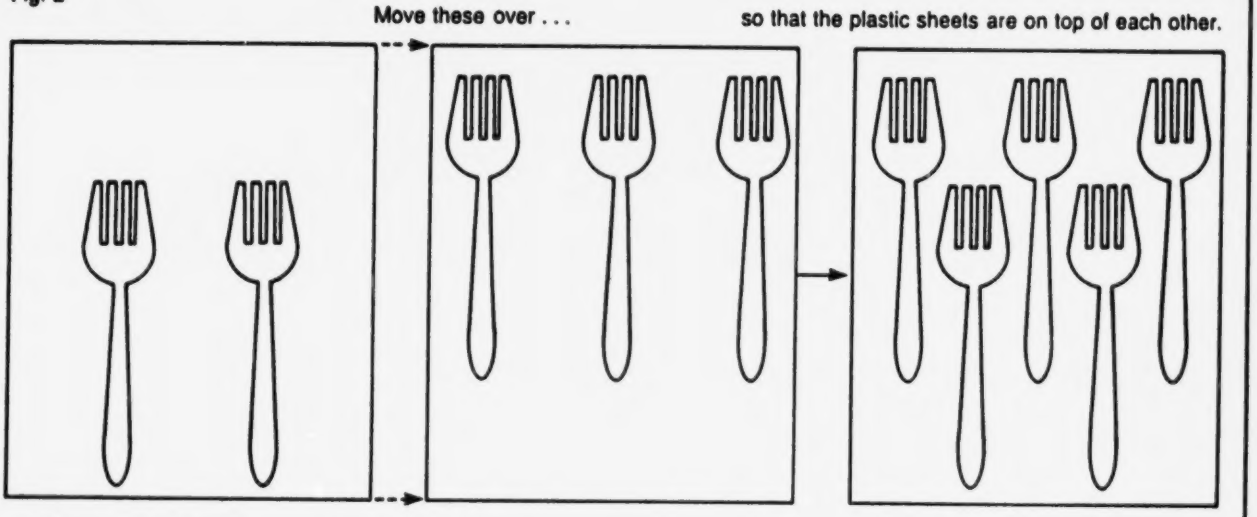
Fig. 2



by being told about mathematics; children must actively participate in thinking mathematically. Teachers need to ask crucial questions that guide children to think through the mathematical concepts being studied. Questions asked by teachers can reveal new directions of thought, encourage children to continue their current line of thought, or provide clues that will stimulate thinking when progress has temporarily been blocked.

Children also need to be given opportunities to verbalize their thought processes; verbal interaction with peers will help learners clarify their own thinking. The development and verbalization of thought processes is time-consuming; the critical variable, however, is not time—it is thinking.

Bridging the gap with activities

Activities involving pictures of objects, textbook illustrations, models on the overhead projector, and drawing, as well as demonstrations by teachers and peers, can smooth the transition between concrete and abstract functioning.

Mathematical concepts emerge from perceptually unrelated ideas that are generated as the learner participates in concrete experiences. The bridging of the gap is the result of a process of internalization. The learner consciously builds on many concrete experiences and emerges with mathematical concepts at the abstract, symbolic level. Teachers must be the catalysts who assist students with this process.

Thought-processing skills, such as the use of logical thinking to understand similarities and differences, can help children bridge the gap from the concrete to the abstract. Thought-processing skills can be developed with such materials as attribute blocks or people pieces. Primary level mathematics students should have many experiences in observing and verbalizing the characteristics of the selected materials. For example, children should be asked to compare two attribute blocks. Then all the attribute blocks can be placed in front of the children, who then sort them so that all the blocks in one pile are alike in some way and all the blocks in the second pile are alike in another way. Children should be encouraged to think of other ways of sorting the blocks. For example, blocks with corners could be placed in one pile and blocks with no corners could be placed in a second pile; or, all the red blocks could be placed in one pile and all the blocks that are not red in a second pile. The same principle can be used to separate the blocks into three piles, then four piles, and so on. These games can enhance students' thinking skills. Looking for patterns and extending them in this manner are excellent ways of developing thinking techniques. An examination of patterns should also be done using number sequences and the place-value system.

A specific example of bridging the gap

Teaching place value is often done at an abstract level, with children using only printed symbols. An abacus with beads of different colors representing different place values is a way of presenting place-value concepts to children on the semiabstract level. The abacus is almost as abstract as place value itself. To bridge the gap we must start on the concrete level and move to the abstract level.

The teaching of place value at the concrete level should begin with blocks or with the actual bundling of ten counters into a package to be moved from the ones place to the tens place. Packages of gum with ten sticks in a package are excellent models to use in teaching grouping by tens. Pictures of blocks or of bundles of ten counters would be appropriate to

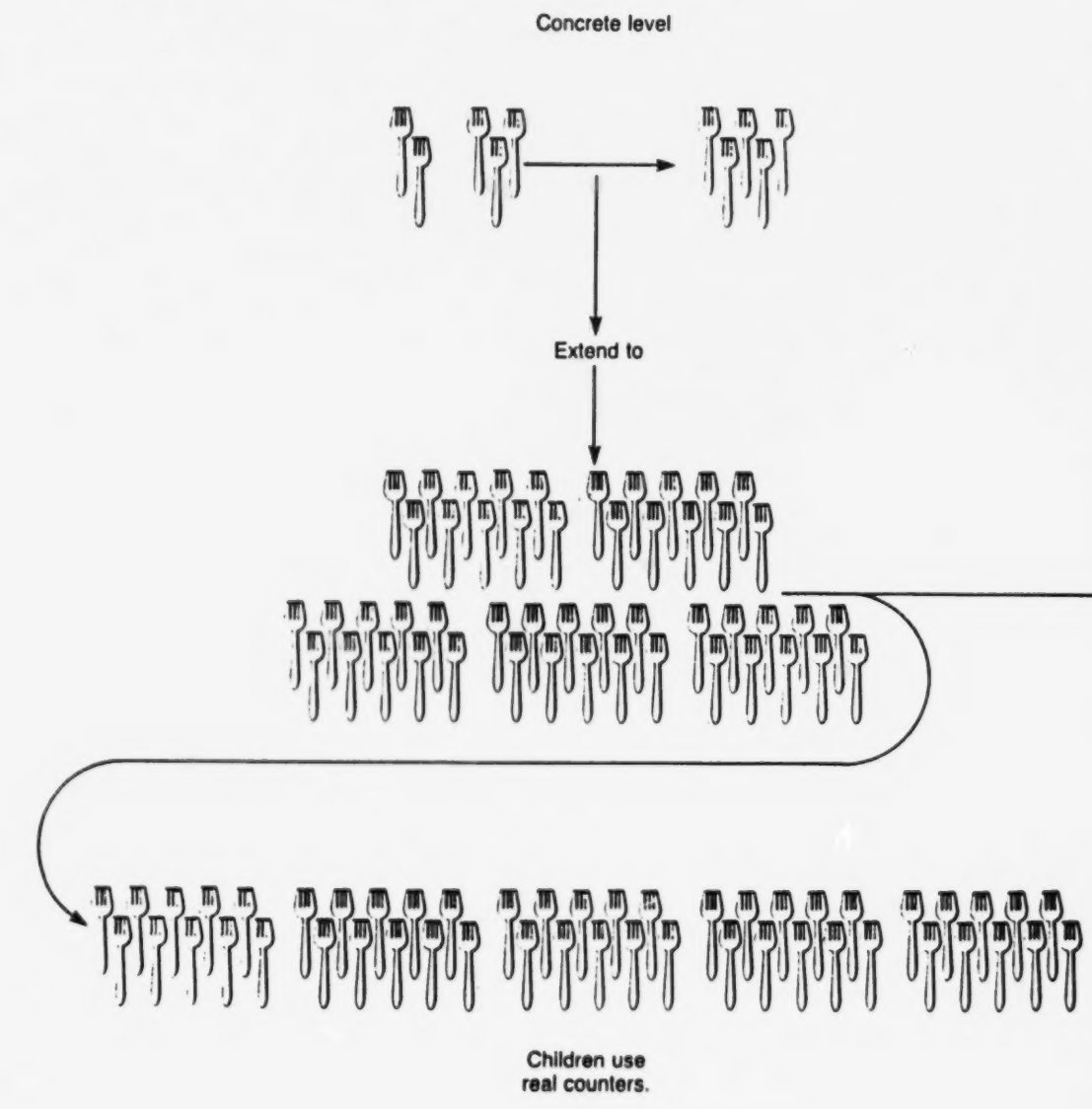
move to the semiconcrete level. The next level, semiabstract, could involve the use of tallies in a place-value chart. All these activities would lead to the abstract level—using symbols to represent place value.

The first step in teaching the basic operations with whole numbers occurs at the concrete level. Children must manipulate things to develop a model for each operation. For example, sheets of paper with plastic forks on them can represent discrete sets; then the two sets can be moved together to obtain a sum for the addition model (see fig. 2).

It is important that children not make a third set to represent the sum; they must place the two disjoint sets together to obtain one set that represents the sum. Demonstrations in which teachers or peers use a variety of models can aid in a student's movement along the continuum. Demonstrations with children would be at the concrete level. The semiconcrete level would involve using pictures of objects. The semiabstract level would involve tallies on the chalkboard or an overhead transparency. The abstract level is reached when the mathematical concept is recorded in a mathematical sentence, such as $2 + 3 = 5$.

Figure 3 illustrates how the gap can be bridged in teaching place value. The concept is extended from the ones place to the tens place. This figure shows how the place-value and addition concepts are brought togeth-

Fig. 3



er to develop meaning for addition of multiples of tens. This idea can now be extended to the addition of ones and tens.

Teachers need to orchestrate mathematical concept development very carefully to provide a smooth transition from the concrete to the abstract levels. The need for a careful sequencing of activities to lessen students' dependence on the concrete level and increase their facility with the abstract level is crucial.

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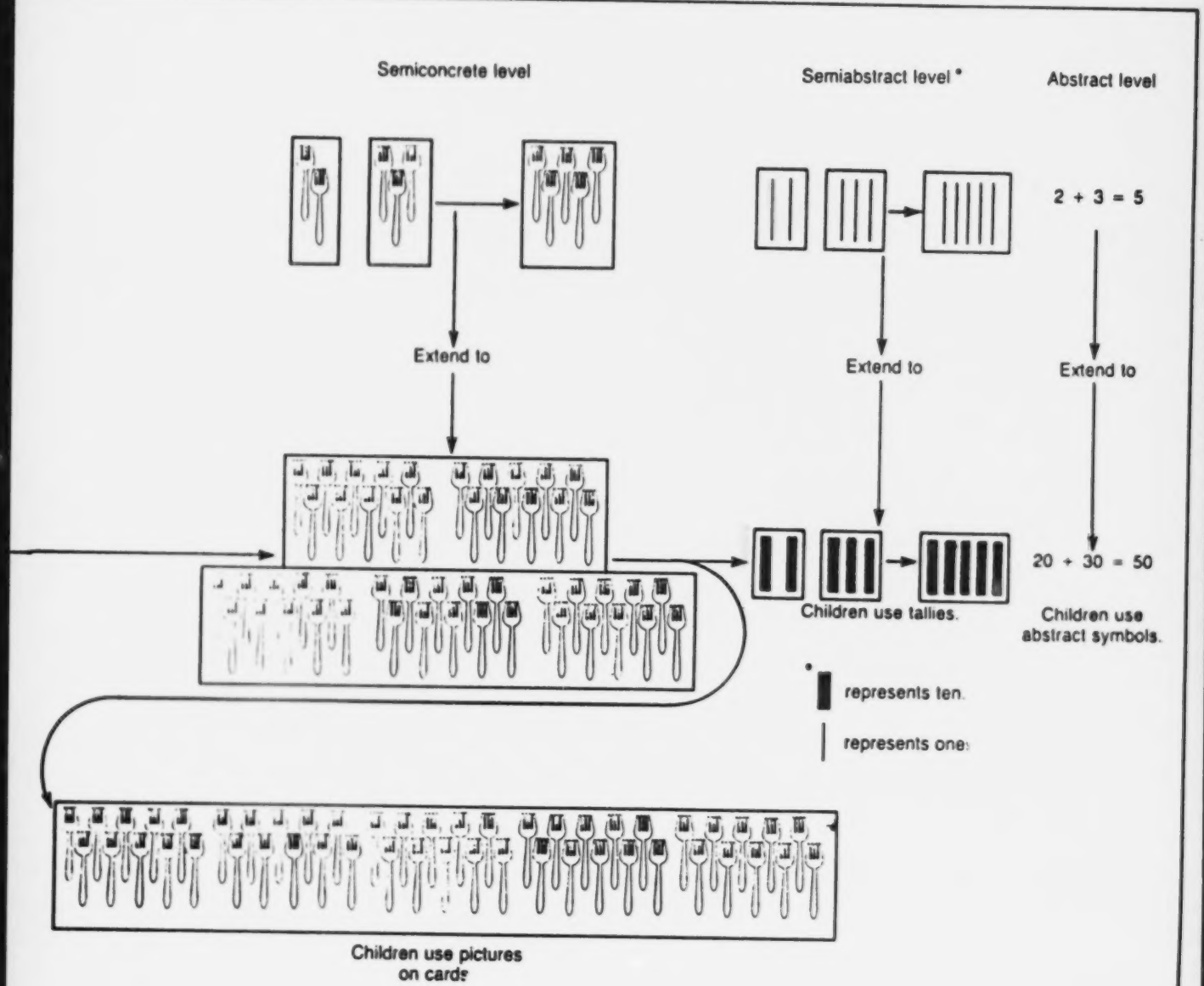

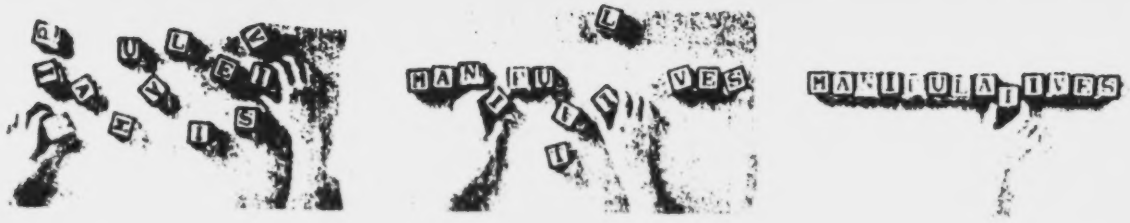

A Rationale

By **Leonard M. Kennedy**

Many teachers believe very strongly in the benefits of manipulative materials in building a foundation for children's mathematical concepts: "I am firmly convinced that manipulative materials improve children's understanding of the mathematics they learn. My classroom is well stocked with all sorts of commercial and teacher-made materials. My children and I use them nearly every day." Other teachers view them with skepticism and are less inclined to use them.

Classroom teachers see the value of manipulatives once they have used them with their classes.

"There just isn't time. And I'm really not convinced that they are important, anyway. They are a lot of bother." Yet strong evidence supports the use of manipulative materials to help children build a firm understanding of mathematical concepts.

First, what are manipulative materials? Manipulative materials are objects that appeal to several senses and that can be touched, moved about, rearranged, and otherwise handled by children. Young (1983, 12) defines these materials as "objects which represent mathematical ideas that can be abstracted through physical involvement with the objects." They can be objects from the environment, such as money or measuring instruments, or materials specifically designed to teach mathematical concepts, such as base-ten blocks and balances.

Support from Learning Theorists

Research on learning provides information that teachers, curriculum planners, and textbook writers use to develop mathematics programs. The mental-discipline and stimulus-response theories of the nineteenth and early twentieth centuries gave way to meaning theory, espoused by William Brownell in the 1930s. This theory is based on the belief that children must understand the basic concepts that underlie what they are learning if learning is to be permanent. Brownell's discussion of learning generated interest in having children use manipulative materials to form the concepts necessary in learning mathematics.

More recent studies of human learning by Jean Piaget (1952) and Richard Skemp (1982) led them to conclude that all individuals pass through stages as they mature. Piaget believed that cognitive development occurs in four stages. Manipulative materials are significant learning aids in all four stages. Students' mental images and abstract ideas are based on their experiences. Hence, students who see and manipulate a variety of objects have clearer mental images and can represent abstract ideas more completely than those whose experiences are meager. Skemp (1982) believes that a two-level director system governs learning. At the first level the manipulation of objects in and out of the school environments provides the learner with physical activities that form the basis for further learning and the internalization of ideas. At the second level, activities are built on those of the first and are mental rather than physical.

Dienes (1960) advocates the use of manipulative materials by children. He describes the use of multibase blocks and other materials to build children's understanding of numbers. Each of the devices supplies a proper concrete representation of a concept. He advocates the use of "multiple embodiments" rather than a single representation of a concept. The multiple representation of place value occurs when children use loose and bundled sticks, bean sticks, ten frames, abacuses, and other place-value materials to learn the meaning of the Hindu-Arabic numeration system.

The strength of the arguments of these and other theorists for using manipulative materials is persuasive. Learning theories suggest that children whose mathematical learning is firmly grounded in manipulative experiences will be more likely to bridge the gap between the world in which they live and the abstract world of mathematics. The manipulatives help children understand both the meanings of mathematical ideas and the applications of these ideas to real-world situations.

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Support from Research

Much research into manipulative materials and their benefits for children has been conducted over the years. Although no single study validates the claim that children should use manipulative materials as they learn mathematics, the collective message garnered from many studies is that the materials are worthwhile.

Fennema (1972) summarized research on Cuisenaire rods versus a traditional approach. The research favored the concrete materials in first grade for the total range of work done in that grade. The value of the rods for second and third graders was less conclusive. She concluded that "there is some indication that children learn better when the learning environment includes a predominance of experiences with models suited to the children's level of cognitive development" (Fennema 1972, 637). She also summarized studies about other materials and stated that "these data tend to support the hypothesis that a learning environment embodying representational models suited to the developmental level of the learner facilitates learning better than a learning environment that ignores the developmental level of the learner" (Fennema 1972, 637). Fennema supports the use of many materials at the early levels of learning, with a gradual decrease in their use as children are able to handle concepts more symbolically.

Research supports the use of manipulatives at all school levels.

Manipulative materials are more widely used in the primary grades than in the upper grades of elementary school. However, research supports their use at all school levels. According to Driscoll (1981), manipulatives not only help children in the intermediate grades develop new concepts but can also be used to provide remedial help. "If there is any risk related to the use of manipulatives in these grades, it derives from their being ignored or abandoned too quickly" (Driscoll 1981, 24).

Suydam (1984) says that lessons that incorporate manipulative materials effectively have a higher probability of producing greater learning than lessons not incorporating such materials. She states that achievement is enhanced across a variety of topics and at every grade, achievement, and ability level (Suydam 1984, 27). The mathematical concepts presented to children in the middle grades are just as abstract and difficult to understand as those presented to younger children. Advanced work with whole numbers, common and decimal fractions and operations on them, measurement concepts, geometry, and many other topics that middle graders deal with are all better understood when children use manipulative materials in appropriate ways.

Support from the Classroom

Each issue of the *Arithmetic Teacher* contains several articles that feature uses of manipulative materials. Most of the articles are prepared by mathematics educators who are convinced of the value of manipulative materials for children. Some are prepared by teachers who know the benefits of a particular material for their students and who write to share their enthusiasm with others.

A middle school teacher, Herbert (1985, 4) wrote that manipulatives allow teachers to create situations that draw mathematical responses from children. Such situations result "in improvements in motivation, involvement, understanding, and achievement—overwhelming reasons to believe that manipulatives are good mathematics."

Sullivan (1981) describes a series of activities that involve play money. She concludes that the materials can be applied to work with percentages, decimal and common fractions, and number bases to help children better understand these areas of mathematics. She also cites the motivational value of money used as a learning aid.

Three sets of materials for teaching rational numbers in the middle grades are described by Jacobson (1984). The models can easily be constructed by children themselves and offer a basis for understanding decimal fractions.

The *Mathematics Their Way* program developed by Baratta-Lorton (1976) places heavy emphasis on manipulative materials. This program has enjoyed success in many classrooms across the country. In the *Mathematics the Piaget Way* program (Hendrickson 1981) used in the Tower-Soudan Schools in Minnesota, manipulative materials also serve as the basis for concept development in mathematics.

The experiences of individual teachers and school systems who have developed programs that regularly involve manipulative materials demonstrate conclusively that such programs can be used at all school levels. Equally important is the demonstration of the fact that manipulative materials do enhance the learning of mathematics.

Concluding Statement

Learning theories and evidence from research and classroom practice support the use of manipulative materials to help children learn and understand mathematics. Well-chosen and properly used manipulative materials enhance children's learning, generate interest, relieve boredom, and promote problem-solving and computational skills.

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(Continued on page 32)

Selection Criteria

By **Michael C. Hynes**

The task of selecting materials for classroom instruction is a crucial one whether the decision involves textbooks (NCTM 1982), software (NCTM 1984), or other teaching aids. This article focuses on a variety of criteria for the selection of manipulative materials. Let's first begin with a definition:

> Manipulative materials are concrete models that incorporate mathematical concepts, appeal to several senses, and can be touched and moved around by students.

This definition implies that students will have materials in their possession to manipulate (Cathcart 1977). It is not sufficient for students to observe a demonstration of the use of an aid. The act of manipulating the aid allows students to experience the patterns and relations that are the focus of mathematics (Adler 1966).

Little disagreement seems to exist among mathematics educators that experiencing ideas on the concrete level is very important to the learning of mathematics, since mathematics deals with abstractions. A major goal of mathematics instruction is to help children learn to operate efficiently at the abstract-symbolic level with an understanding of the concepts or skills in question. The purpose of using manipulatives is to assist students in bridging the gap from their own concrete environment to the abstract level of mathematics (Fennema 1973). It is the responsibility of the teacher to use these concrete models to foster understanding of the abstract, relate pictures of the models to aid in the movement toward the abstract, and provide practice at the abstract level to insure mastery of a concept or skill. Clearly, manipulative materials are a means to an end (Cathcart 1977; Reys 1971; Sowder 1976). The task of selecting manipulatives becomes a response to the question, "What are the important characteristics of effective manipulative materials?" The following pedagogical and physical criteria should be considered when selecting manipulative materials for a particular purpose.

Pedagogical Criteria

1. *Clear representation of mathematical ideas*

Manipulatives can help children understand mathematical concepts only if the material selected clearly represents the idea in question. The representation of an idea should be natural and easily understood (Sowder 1976). Many contrived models of multiplication of integers have appeared in the literature, but few, if any, manipulative materials offer a student a clear picture of why a "negative times a negative is a positive."

If the goal is to teach subtraction with regrouping, materials that can actually be regrouped are mandatory.

2. *Appropriateness for students' developmental level*

It is important to select materials so that all children can learn using an aid appropriate for their developmental level and learning style (Reys 1971). A critical concern is providing for the wide diversity of individual differences among students (Fennema 1973). Some students learn best while attending to a particular attribute of a manipulative, say length, whereas others may not grasp the significance of the relative length of rods, rows of cubes, or sticks. Attribute blocks usually include the attributes of size, shape, color, and thickness. Cuisenaire rods capitalize on the use of color and length; the focus of the geoboard is primarily on length and area. It is important that manipulatives be selected on the basis of the children's ability to operate effectively with the apparent attributes of the aid.

Materials selected for use must also be consistent with the motor development of the child. For example, a young child may have difficulty manipulating such materials as centimeter cubes, which are very small. The use of a larger cube can be more appropriate in such a situation.

3. *Interest*

Manipulative aids do arouse students' interest and improve their motivation to learn mathematics (Fennema 1973). This motivation is often elicited by the physical characteristics of the materials that will be discussed later. Furthermore, interest seems highest

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Table 1
Manipulative Material Checklist

Manipulative: _____ Supplier: _____ Cost: _____

Concepts developed or reinforced _____

Grade levels: _____

	Poor	Below average	Average	Above average	Excellent
Pedagogical criteria					
1. Clear representation of mathematical ideas					
2. Appropriateness for students' developmental level					
3. Interest					
4. Versatility					
Physical criteria					
1. Durability					
2. Simplicity					
3. Attractiveness					
4. Manageability and ease of storage					
5. Reasonableness of cost					

when each child can take part in the lesson by manipulating the aids.

4. *Versatility*

Some materials stimulate a number of senses at the same time and can be used to teach many mathematical concepts at various grade levels. For example, many materials can be used to teach whole-number, rational-number, and geometric concepts. Grossnickle et al. (1983, 37–39) provided some useful information on the versatility of several manipulatives: abacus, attribute blocks, base-ten blocks, geoboards, pattern blocks, Unifix cubes, and place-value pocket charts.

Physical Criteria

Pedagogical concerns are important in the selection of manipulative materials, but a consideration of the physical characteristics of manipulatives is also crucial. Information about the physical appearance of materials is often available in reviews in the *Arithmetic Teacher* and in commercial catalogs and brochures. In reading this literature, you should note the durability, simplicity, attractiveness, ease of manageability, and storage requirements, and, of course, the cost of the materials.

1. *Durability*

Manipulatives are often costly investments. Thus, the materials should be made to withstand normal use and handling by children (Reys 1971; Grossnickle et al. 1983). Materials that seem too fragile for use by children are of little value. Remember, it is imperative that students actually manipulate the materials.

2. *Simplicity*

A manipulative should be easy to operate. The complexity of its construction should not distract or confuse a student. Because of the desire to buy aids that are multipurpose, care must be exercised to select simple devices that clearly portray concepts being taught.

The simplicity of a manipulative is directly related to the complexity of the mathematical idea being modeled. It seems, however, that the simpler models of even more complex ideas are the ones most easily remembered (Dewdney 1985).

3. *Attractiveness*

Attractiveness is one key to increasing students' motivation and interest. The manipulatives should be bright, precisely constructed, and aesthetically pleasing (Hamilton 1966; Reys 1971). The appearance of the aid should not distract the student from the concept to be learned, however. Periods of free play with the manipulatives, before mathematics instruction, can help insure that students are attentive to the modeling of mathematics and not the aid itself.

4. *Manageability and ease of storage*

Careful plans must be made for the distribution and collection of materi-

Modeling Operations and Algorithms

By **Ian D. Beattie**

The idea of using manipulative materials to model operations and algorithms is not new. The *Arithmetic Teacher* frequently includes articles on the topic, and its readers should have accumulated a wealth of excellent teaching suggestions. Current methodology books and school texts place great emphasis on the use of manipulatives. Thus one might expect appropriate manipulative materials to be a mainstay of every mathematics classroom, but evidence suggests otherwise. Although teachers *believe* that manipulative materials should be used for mathematics instruction and that the use of such materials *does* develop understanding, the use of manipulative materials diminishes through the grades (Suydam 1984a).

Several reasons for this situation might be put forward. First, each operation has several interpretations or meanings, but only one standard algorithm is taught. In the solution of word problems, once the appropriate operation is chosen, the algorithm is performed without reference to the interpretation of the operation embedded in the word problem. That is, the algorithm becomes a rote mechanical procedure. In fact, after the algorithm is understood, it *should* be done by rote. One should not have to think through every simple computation. This rote notion of an algorithm is reinforced by pages of computations that *have* no reference to any particular interpretation and by the fact that this method is how teachers themselves *learned* the algorithms. This situation leads to the second reason.

Preservice teachers and many experienced teachers have great difficulty making the connections among the meanings of an operation, the associated manipulation of materials, the verbal explanations that accompany the manipulations, and the steps of the algorithm.

Third, teachers are pragmatic. Achievement is measured by how well students *do* the algorithms, not how well they understand them, and children must have adequate mastery of facts and algorithms if they are to be successful in later grades. Consequently, teachers focus instruction on mastery of the mechanics of algorithms and not on understanding. With the increasing emphasis on mastering algorithms in the upper grades, the decrease in use of manipulatives is most likely inevitable. Probably inevitable, too, is the type of meaningless explanation that follows. It was given by a sixth-grade remedial student, and it is an extreme, but not uncommon, example of rote explanation:

Manipulative materials are the key to understanding operations and algorithms.

$$\begin{array}{r} {}^{4}\not{5}\ {}^{11}\not{7}\ {}^{1}8 \\ -\quad 3\ \ 9 \\ \hline 4\ \ 8\ \ 9 \end{array}$$

"9 doesn't go into 8. Borrow from 2, cross out 2, put 1 there and 1 there. 9 from 18 is 9. 3 doesn't go into 1, cross out 5, put 4, another 1 beside 1, equals 11. 3 take away 11 is 8, bring down 4."

Those who work in the area of diagnosis and remediation come across many such examples of rote procedures and inappropriate language. The use of such terms as *borrow*, *carry*, *bring down*, and *cross out* is common across all ability levels.

A great deal of time is spent on teaching the standard algorithms for whole and rational numbers. Usiskin (1983) is probably correct when he writes, "A century from now people will gasp when they learn that six to nine years were spent getting good at pencil and paper arithmetic. They will ask: Didn't they have calculators then?" Does this mean that the teaching of algorithms should be discontinued? No, but it does mean that the *reason* for teaching algorithms should change from obtaining correct answers through the rote manipulation of symbols to understanding the meaning of the operation and the rationale behind each step in the algorithm. And the *way* in which algorithms are taught should change from a mechanical procedure to methods that will facilitate *understanding*. This understanding will become more, not less, important in the calculator and computer age.

Few elementary school teachers would wish to explain the addition algorithm as did Dodge (1969):

To add 23 and 48 we write

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$$
\begin{aligned}
23 + 48 &= (2 \cdot 10 + 3) + (4 \cdot 10 + 8) \\
&= (2 \cdot 10 + 4 \cdot 10) + (3 + 8) \\
&= (2 + 4)10 + 3 + 8 \\
&= 6 \cdot 10 + 11 \\
&= 6 \cdot 10 + (1 \cdot 10 + 1) \\
&= (6 \cdot 10 + 1 \cdot 10) + 1 \\
&= (6 + 1)10 + 1 \\
&= 7(10) + 1 \\
&= 71
\end{aligned}
$$

. . . We have obtained 23 + 48 = 71 in a most tedious manner. How much easier to write:

$$
\begin{array}{r}
23 \\
+\ 48 \\
\hline
71
\end{array}
$$

But why can it be used? What justifies it? . . . The long form for addition explains the short algorithm. It demonstrated why the short form works and how its carrying is justified.

Tedious indeed! But Dodge's explanation does stress that an algorithm is more than a rote procedure. It is a systematic, iterative application of the basic properties of our numeration system.

Evidence shows that teaching for understanding of operations and algorithms will help computational mastery. In reviewing research on fractions, Suydam (1984b) concluded that "we need to shift the emphasis from having students learn rules for operations on fractions to helping them develop conceptual bases for fractions." Again dealing with fractions, Driscoll (1984) observed that "once the procedures for computing fractions are forgotten, few teenagers can reconstruct them," that "to use rational numbers a child must understand rational numbers," and that "all levels of students need experiences with concrete materials." The dramatic improvement of computational scores obtained by Wheatley (1984) in a study that emphasized problem solving and manipulative materials and deemphasized computation offers further support for the use of manipulative materials. Thus it appears that whether a teacher's goal is computational mastery or understanding, manipulative materials are important.

One might go further and argue that manipulative materials are the key to understanding operations and algorithms. In an excellent article Young (1983, 12) defined manipulative materials as "objects which represent mathematical ideas that can be abstracted through physical involvement with the objects." It is important to note that mathematical ideas are abstract, but the nature of human growth and development demands pedagogical approaches that involve representations of these ideas. Rousseau (1985) discussed the role of four kinds of representations: (1) concrete representations, which are three-dimensional and manipulable, such as base-ten blocks; (2) pictorial representations, such as number lines, graphs, and pictures of blocks; (3) language representations, which are the accepted oral and written vocabulary that has been given specialized meaning; and (4) alphanumeric representations, such as 2, +, 7, and the standard algorithms. When the teaching emphasis is placed on the fourth class of representations, algorithms are taught as rote procedures, language is inappropriately used, and, most important, understanding is lacking. The teaching emphasis should be on the connections among the meanings of an operation, the associated manipulation of materials, the verbal representations that accompany the manipulations, and the steps of the algorithm. Manipulative materials are the keys to making those connections in four important, distinct, but closely related, ways.

An understanding of algorithms will become more important in the calculator and computer age.

1. *Materials clarify the concept or meaning of each operation.* For each operation several different real-life situations are solved by application of that algorithm. Addition is applied to joining sets or to counting on; subtraction is appropriate for take away, comparison, missing addends, or partitioning of a set into two subsets; multiplication is appropriate for joining equivalent disjoint sets, for row-by-column, area, or Cartesian product situations; and division is applied when sharing out (partitive) or repeated subtraction (measurement) is called for. However, although the same algorithm is applied in each instance, the physical situations are different, and manipulative materials are essential for showing these differences.

2. *Materials clarify the language of each operation and algorithm.* As illustrated by the preceding subtraction example, a great deal of "gibberish" is associated with the algorithms. The description given has no relationship either to place-value concepts or to the physical manipulations that would be appropriate in arriving at a solution. Words like *borrow* and *carry* cannot be justified as other than rote verbal memory aids, and such words promote errors when students attempt to apply them to nondecimal situations, as in the subtraction of fractions when regrouping is necessary. The use of "minussing" and "timesing" for subtraction and multiplication, respectively, and "goes into" for division can hardly lead to understanding. Verbal descriptions *are* necessary, but they can and should be mathematically appropriate.

3. *Materials clarify the algorithm.* As mentioned earlier, algorithms are more than rote procedures; they are based on the positional notation and basic properties of our numeration system. Appropriate manipulative materials show these properties in action, and each step of an algorithm describes some action on the materials. An understanding of the algorithm is built by associating the steps of an algorithm with the actions on the manipulative materials.

4. *Materials clarify pictorial representations.* Textbooks rely heavily on the communication of ideas through pictures, but as Campbell (1981) has shown, the ability to interpret pictures is developmental—children have to learn how to interpret pictures. By modeling operations and algorithms with objects, children will be better prepared to interpret pictures that model operations and algorithms.

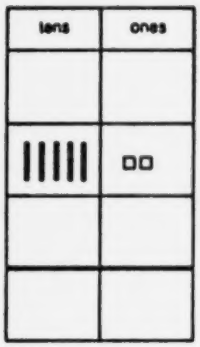
Different physical situations can model a given operation, but the standard algorithm is identical for each situation. For example, the several interpretations of the operation of

Fig. 1 Correlating manipulatives, language, and algorithm for two models of subtraction

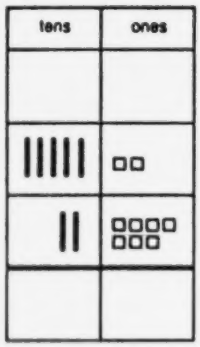
Problem A. "Jim had 52 marbles but lost 27. How many does he have left?"

Problem B. "Jim has 52 marbles. John has 27 marbles. How many more marbles does Jim have?"

1.



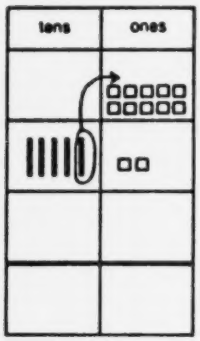
"Show 52, putting ones and tens in the correct columns."



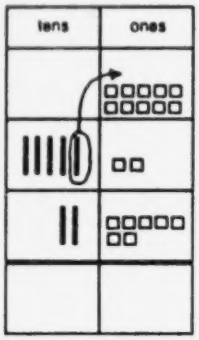
"Show 52 and 27, putting ones and tens in the correct columns."

$$\begin{array}{r} 52 \\ -27 \\ \hline \end{array}$$

2.



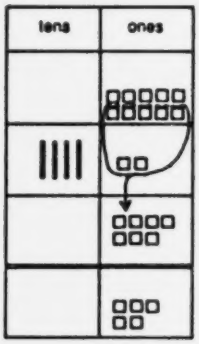
"You can't take 7 from 2. Regroup a ten into ten ones. This gives four tens and twelve ones."



"Compare the ones. 7 is greater than 2. Regroup a ten into ten ones. This gives four tens and twelve ones."

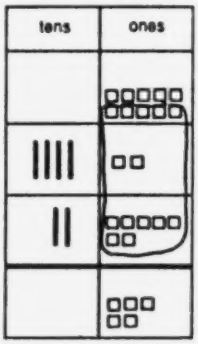
$$\begin{array}{r} {}^{4}\,{}^{1} \\ 5\,2 \\ -2\,7 \\ \hline \end{array}$$

3.



"Take away seven ones."

"Show how many are left."

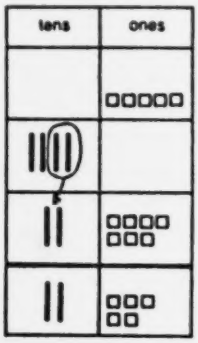


"Match up the ones."

"Show the difference."

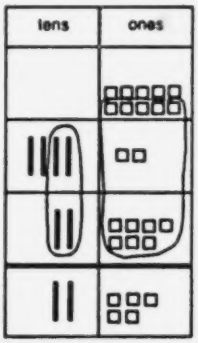
$$\begin{array}{r} {}^{4}\,{}^{1} \\ 5\,2 \\ -2\,7 \\ \hline 5 \end{array}$$

4.



"Take away two tens."

"Show how many tens are left."



"Match up the tens."

"Show the difference."

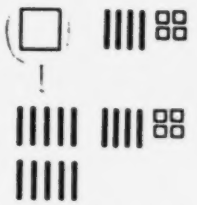
$$\begin{array}{r} {}^{4}\,{}^{1} \\ 5\,2 \\ -2\,7 \\ \hline 2\,5 \end{array}$$

Fig. 2 Correlating manipulatives, language, and algorithm for the partitive model of division

Susan has a package of 144 candy canes that she wants to share evenly among six friends. How many canes will each friend get?

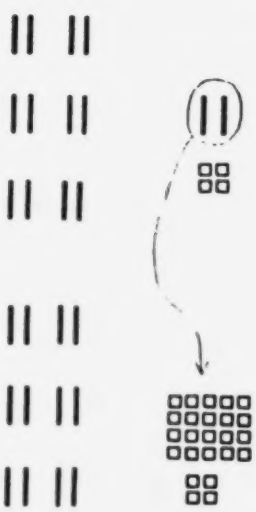
Show the 144 canes to be divided evenly among 6 friends.

$6\overline{)144}$



You can't give a hundred to each. One hundred is ten tens. So you have fourteen tens altogether.

$6\overline{)14|4}$

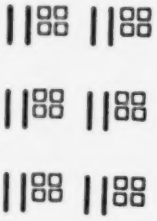


Give two tens to each friend and record 2 in the tens column. This partitioning takes twelve tens, leaves two tens.

$$\begin{array}{r} 2 \\ 6\overline{)14|4} \\ \underline{12} \\ 2 \end{array}$$

Each ten is ten ones. With the four left, this makes twenty-four ones.

$$\begin{array}{r} 2 \\ 6\overline{)14|4} \\ \underline{12} \\ 2|4 \end{array}$$



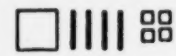
Give four canes to each friend and record a 4 in the ones column. This sharing takes twenty-four canes, and no canes are left.

$$\begin{array}{r} 24 \\ 6\overline{)14|4} \\ \underline{12} \\ 2|4 \\ \underline{2|4} \\ 0 \end{array}$$

So each friend got 24 canes.

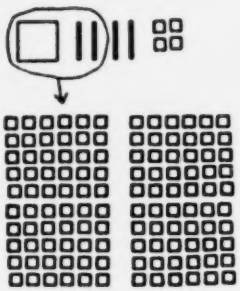
Fig. 3 Correlating manipulatives, language, and algorithm for measurement division

Susan has a package of 144 candy canes, enough for her to give 6 to each of her friends. How many friends can she share the candy canes with?



Show the 144 canes to be divided into groups of 6.

$$6\overline{)144}$$



You can't make a hundred groups of 6.

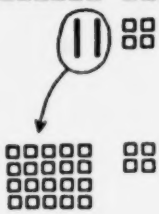
You *can* make twenty groups of 6. Twenty is two tens.

Record 2 in the tens column.

Take one hundred and two tens away.

Two tens are left.

$$\begin{array}{r} 2 \\ 6\overline{)144} \\ \underline{12} \\ 2 \end{array}$$



Each ten is ten ones. With the four left, this makes 24 ones.

$$\begin{array}{r} 2 \\ 6\overline{)144} \\ \underline{12} \\ 24 \end{array}$$

Make another four groups of 6. Record the 4 in the ones column. Take the 24 away.

$$\begin{array}{r} 24 \\ 6\overline{)144} \\ \underline{12} \\ 24 \\ \underline{24} \end{array}$$

There are 24 groups of 6 all together. So Susan can share with 24 friends.

subtraction are all solved by applying the standard subtraction algorithm. Manipulative materials model these situations, but actions on the materials, the verbal descriptions of the actions, and the steps of the algorithm must be coherent. What one *does*, what one *says*, and what one *writes* must match. Figure 1 shows how this matching can be done with two models of subtraction. The algorithms are identical, but the physical manipulations and associated explanations are different. The reader must, of course, infer the manipulations. The explanations can vary, provided they emphasize "taking away" for problem A and "comparing" for problem B.

Similarly, figures 2 and 3 show that for two models of subtraction, the algorithms are identical but the manipulations and explanations differ.

One final point can be made. A reduction in the amount of time spent on working with manipulative materials may have a short-term payoff in students' computational facility, but it minimizes their potential for transfer. It may be that many of the difficulties children encounter in such areas as fractions or decimals occur because children have learned rote procedures without fully understanding the operations or the algorithms for whole numbers.

Summary

Manipulative materials can play an important role in learning operations and algorithms. They can clarify the several interpretations of each operation, establish a basis for correct mathematical language, and show why algorithms work. Manipulatives can make sense out of nonsense.

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Special Learners

By **Carol A. Thornton** *and* **Barbara Wilmot**

Children with special learning needs fall into two distinctly different groups: the learning handicapped and the mathematically gifted. In this article consideration is given to the important role of manipulatives in mathematics instruction for each of these groups of students.

Using Manipulatives with the Learning Handicapped

Classroom teachers today work with larger proportions of handicapped students in their mathematics classes than ever before. Probably the biggest single obstacle for the learning handicapped is their inability to organize information to be learned. Developmental lags, poor retention, tendencies to be easily distracted, perceptual problems, logical thinking or linguistic difficulties—any of these may account for the inability to organize. Each student is different, but most have one trait in common: they benefit from a carefully structured, active approach to learning mathematics.

As handicapped learners are guided in their work with materials and are given the chance to *verbalize what they do*, they begin to internalize and make sense of things in their own minds. Structured situations involving manipulatives help these students organize their thinking so that they begin to see relationships or follow the flow of a computational procedure. Four ideas for providing this structure follow.

1. *Vary the way you present manipulatives in a lesson to capitalize on students' learning strength.* When introducing or summarizing a manipulative activity, stagger the presentation of visual, verbal, and tactile information (Toohey 1984). For example, if a student is a strong visual learner, a silent visual demonstration might be a good introduction to a lesson. Or, for auditory learners, provide a verbal overview of what is to happen. At least initially, students do not have to process visual, auditory, and tactile information at the same time. This approach makes it possible for you to motivate students better and capture their attention at key points in a lesson.

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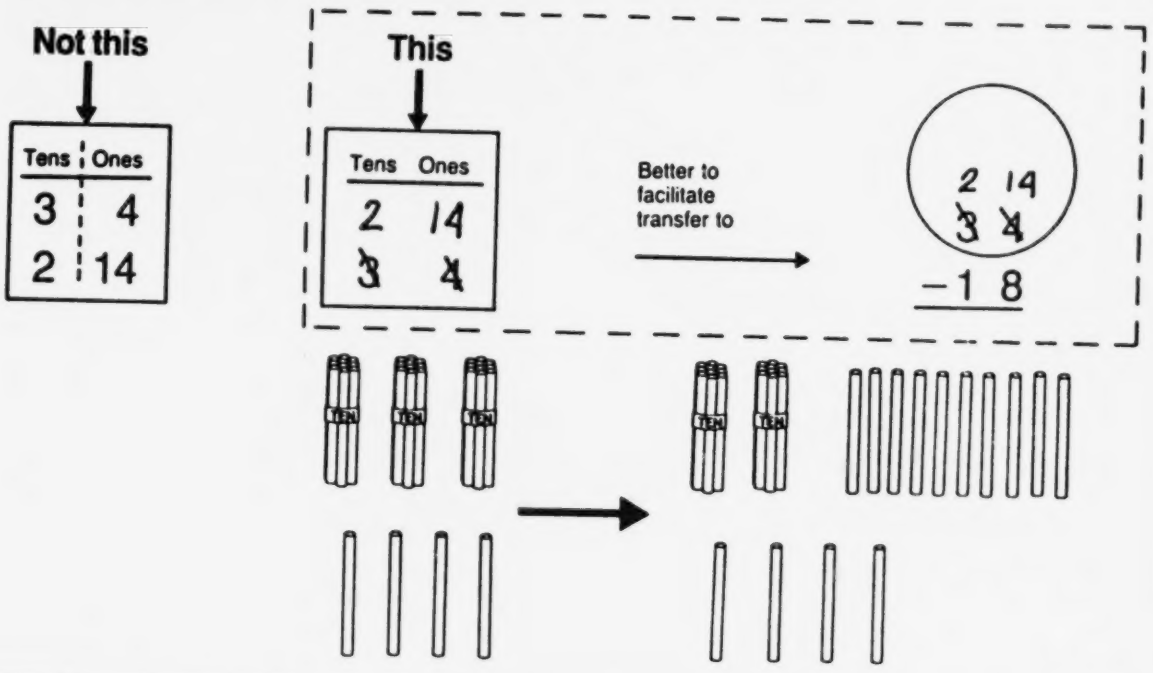
Fig. 1 This example of renaming illustrates an approach that is more effective for the learning handicapped student than the standard treatment offered in textbooks.



2. *Help students build important self-monitoring skills.* In addition to important concepts and skills, learning handicapped students must also be taught *how to learn* (Thornton and Toohey 1985). Central to this perspective is the development of self-monitoring skills, which are critical when students are under pressure (as in tests), are tired, or are under stress. Appropriate questions *repeated* during active instruction with materials, illustrated in the examples that follow (see idea 4), elicit responses during initial instruction. These same questions often are internalized and used later by the students themselves as the basis for selecting correct principles or procedures for completing mathematical tasks.

3. *Provide for consistency.* Often special educators and classroom teachers must cooperate in planning and implementing the mathematics curriculum for handicapped learners. It is critical that these teachers agree on ways to present content to students. Sometimes this may require refining textbook suggestions in prebook activities with manipulatives (see fig. 1).

4. *Carefully plan the use and the* phasing out *of manipulative materials both in concept and in skill learning.* The following six-step teaching sequence has proved helpful in work with most learning handicapped students.

1. Manipulate—talk

Use materials to model an example. Talk through each step. Unless children are highly distractible, involve them almost immediately in the manipulation. Those with attention disorders should watch and interact verbally during the first few examples so that, under the teacher's control, they can better attend to the thinking being highlighted.

2. Manipulate—write

Use manipulatives to model the example. Write up each step as it is completed. The goal now is to relate the manipulation of objects to what is written by having students *verbalize the thinking.* Learning handicaped students often are just told what to do. Since many of these students only internalize ideas when they *hear themselves speak,* they need to be led into describing procedures orally in order to learn them.

Example

34
18

Use bundles of tens and single sticks to model the example. "Four ones. Is that enough to take away 8?" (No, so invite students to trade using sticks, to describe verbally what was done, then to record to show now there are 2 tens and 14 ones.) Continue in a similar manner to subtract ones and then tens.

One must also structure the activity so that students do not become overly dependent on concrete representations. For example, do not allow children to count objects to find an answer to computational examples. If facts aren't mastered, make a fact list available until they can be worked on separately. Put the materials in a box

(Tucker 1983) or cover them with a hand after asking, "How many now?" Then, when an answer is given orally, students can count, *if necessary*, to check

3. Talk—write—manipulate: step by step

Talk through each step, write what you say, and use manipulatives after each step is recorded to check *(five minutes a day for several days)*. This step is the most critical for helping handicapped learners internalize appropriate language and thinking. Too often we stop short of this important step, which clinches initial mastery and fosters retention.

Example. Extend the work begun in step 2. Let students talk through each step (prompt if necessary). Students should now recognize that a trade is needed and should tell you that 3 tens and 4 ones become 2 tens and 14 ones after the trade. They would record this (as in the subtraction problem of fig. 1), then use the sticks to check that what they've written is correct. They then continue subtracting ones and tens in a similar manner.

4. Talk—write—manipulate: — entire solution

Talk through the entire solution, recording as you go. Then model the entire example as a check. Given that attention to *students' talk*, to the *use of materials* to model procedures (step 2), and to checks of written answers (step 3) has been adequate, students typically are ready to carry on independently. At first require them to work each problem to the end, then to use materials to check their solutions. When children themselves can correctly demonstrate and explain the thinking they used to complete an example, then manipulatives are no longer needed.

5. Discriminate

Help students discriminate new from previous learning. For example, have students circle and do just those problems that require renaming when these are mixed with simpler problems with no renaming.

6. Compute—model

From time to time, use manipulatives to check written work. The goal now is maintenance. If this step is neglected, learning handicapped children will narrowly focus on getting answers, make conceptual errors, and fail in the transfer to related learning.

Similar instructional sequences have been developed and used with regular-class students by other mathematics educators (e.g., Tucker pers. com. 1981; Rathmell 1983). For the learning handicapped, however, such approaches are critical to success in mathematics and, therefore, are necessary for effective teaching of these special students. We suggest that verbal problem solving that relates written work to real-life applications be incorporated throughout. Otherwise, many handicapped learners will not make the transfer to using mathematics outside the classroom.

Using Manipulatives with the Mathematically Gifted

Manipulative materials are valuable tools for gifted students. When learning the basic facts and algorithms for the four operations with whole and rational numbers, these special students also need to have experiences at the concrete level. However, the number of examples necessary is usually considerably fewer than that for the other students because most gifted students have good visual imagery and can perceive relationships and make generalizations. For example, when students in a Saturday class of gifted fifth graders complained about long division, two examples (752 ÷ 3 and 409 ÷ 5) using partitive division with the base-ten blocks were adequate to allow them to understand what they were doing and how to record results.

In an enrichment curriculum for gifted students, concrete materials have an important place. Setting specific expectations for their use only limits potential outcomes. The learning of particular skills may be one purpose of an activity, but the overriding goal for gifted students should be to extend their thinking to higher levels, using the physical materials as a tool. The best approach is for the teacher to provide materials and general directions and then to ask appropriate questions (such as those that follow). Students will move in various directions according to their abilities, motivation, and previous experience.

The classroom climate should be informal so that students feel comfortable talking, moving about, and experimenting. Working in groups of two to four offers students an opportunity to interact with peers as they learn to brainstorm, analyze, cooperate, compromise, and evaluate their own abilities. Manipulative aids can be introduced in activities that are paced to meet each student's or group's needs. Telling should be minimized and discovering maximized. The teacher does not need to know the answers to all the questions and indeed should include questions that have no answer.

When used appropriately with gifted students, manipulative aids have at least six functions. They can—

1. build concepts and related vocabulary;
2. improve spatial visualization;
3. allow discovery of patterns and relationships;
4. provide problem-solving experiences;
5. teach the essence of verification or proof; and
6. promote creativity.

The following section lists some important mathematical concepts and terms that can be developed using manipulative aids with gifted students.

Concepts

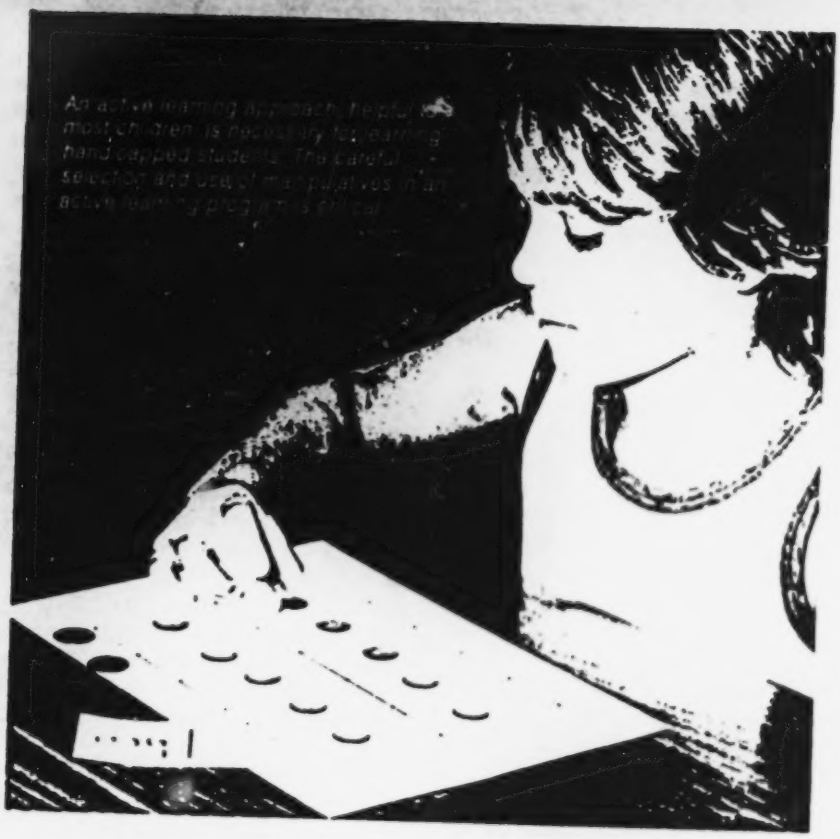
Mira—symmetry, flip, rotation, perpendicular, bisect

Tangrams—congruent, similar, acute, obtuse, right

Cuisenaire rods—prime, composite, factor, multiple

Attribute blocks—set, union, intersection, subset, attribute

Polyhedra—faces, edges, vertices, surface area, volume



The following sections suggest, for several learning aids, a typical question that might be asked to extend the thinking of mathematically gifted students. Actually, an entire unit might evolve around each suggestion.

Spatial visualization

Tangrams—"Can you make a parallelogram that is not a rectangle using four of the pieces? Five? Six?"

Pattern blocks—"Can you make a design with exactly three lines of symmetry?"

Interlocking cubes—"How many cubes would it take to make an empty cube with four blocks forming an edge? Six? Twenty?"

Patterns and relationships

Attribute blocks—"Can you make a circle of ten blocks, each with two value differences from the adjacent blocks?"

Dominoes—"Your set of dominoes has one missing. Can you find it?"

Tangrams—"Make a design. If the square has a value of 1, what is the value of your design?"

Cuisenaire rods—"How many different trains can you make if you have three rods of different colors? Four? Seven? Twenty?"

Multibase blocks—"How could you represent 567 in base four? Base six? Base twelve? Base twenty?"

Problem solving

Tessellations—"Which of the regular polygons will tessellate? Why?"

Attribute blocks—"Can you fill a six-by-six matrix with blocks so that each block has a one value difference from its horizontal and vertical neighbors?"

Pattern blocks—"Each angle of the square measures 90° and each angle of the triangle measures 60°. Can you find the measure of all of the angles in all the pieces?"

Verification

Cuisenaire rods—"Can you show that the sum of the squares of the two legs of a right triangle is the same as the square of the hypotenuse?" "Can you show me that 21 is a factor of 168?"

Geoboard—"Can you show that the area of a trapezoid is one-half of the height times the sum of the length of the bases?"

Mira—"Can you show that this angle has been bisected?"

Creativity

Tessellations and mirrors—"Can you design a linoleum pattern?"

Geoboard—"How many different-sized squares can you represent on a board with 25 nails? 36 nails? 144 nails?"

Pattern blocks—"Design a dodecagon with the blocks. Can you write a computer program in Logo that will duplicate your design?"

Concluding Comments

This article has focused on the use of manipulatives with two groups of special learners: those with learning difficulties and those with special abilities. It has highlighted different choices in concrete aids and, more important, different purposes and procedures for using those aids to nurture the process of learning mathematics in ways each of these groups of students is capable.

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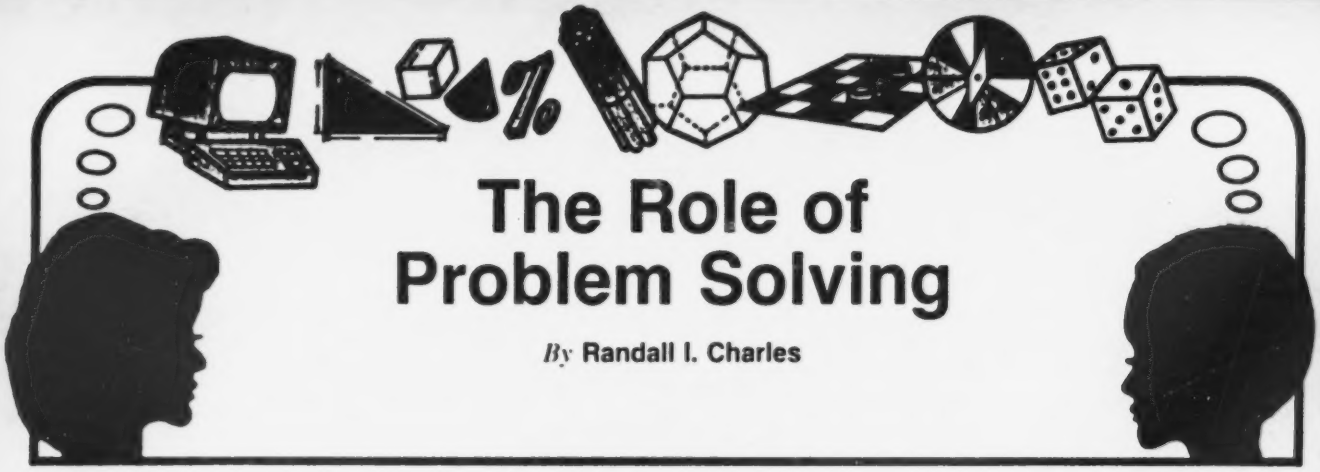
Problem Solving

and

Mathematical Thinking

Resources

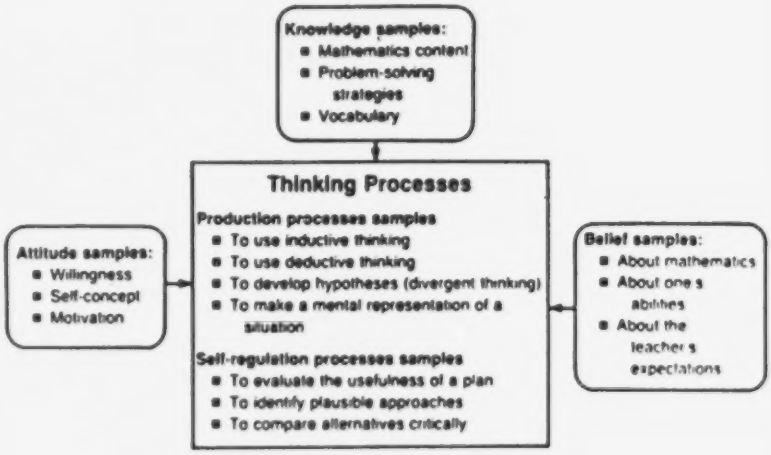


The Role of Problem Solving

By **Randall I. Charles**

The importance of problem solving in mathematics has been attested to by many individuals and groups (e.g., Snowmass 1973; NCSM 1977; CBMS 1982). Furthermore, the belief seems to be common that the development of students' problem-solving abilities is one of the most important goals of mathematics education. In view of the importance of problem solving, it is tempting to argue that problem solving and mathematical thinking are in fact different names for the same activity. However, such an argument would provide too narrow an interpretation of mathematical thinking and too broad a view of problem solving. The purposes of this article are to describe one view of "mathematical thinking" and to describe the characteristics of a problem-solving program necessary to develop this kind of thinking.

Fig. 1 Components of mathematical thinking



What Is Mathematical Thinking?

Mathematical thinking is not well defined. Discussions with several mathematicians will usually result in several different interpretations. Furthermore, a mathematician's view of mathematical thinking is not usually the same as the view of a psychologist, an elementary school teacher, an expert on critical thinking, or a person on the street. Mathematical thinking has many interpretations, and each may be correct or useful depending on one's perspective.

In the elementary school classroom, mathematical thinking involves the coordination of four components:

1. Thinking processes
2. Knowledge
3. Beliefs
4. Attitudes

Figure 1 shows the relationships among these components. The key component, *thinking processes*, consists of *production processes* and *self-regulation processes*. Production processes are mental procedures for representing and acting on information (e.g., concepts, rules, or examples). Self-regulation processes are mental procedures used to monitor, evaluate, and direct one's selection and use of production processes. Self-regulation processes are consciously selected and controlled by the learner. The other components in the model—*knowledge*, *attitudes*, and *beliefs*—are categories of factors that affect one's use of thinking processes. The knowledge component includes previous mathematical knowledge, as well as other knowledge that one has acquired that can be brought to bear on a mathematics learning task. The attitude and belief components contain both debilitative (e.g., "I don't like math!") and helpful (e.g., "Some

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mathematics problems have more than one answer.") attitudes and beliefs.

Problem Solving and Mathematical Thinking

Students encounter a variety of types of learning in the study of mathematics, and each type requires mathematical thinking. Problem solving and the learning of concepts, facts, and algorithms are four examples. *Problem solving has tremendous potential for promoting the development of mathematical thinking.* However, to achieve this potential the instructional program in problem solving must be properly designed and implemented. The following three guidelines for developing and implementing a problem-solving program can promote the development of mathematical thinking. The first deals with the content of instruction, and the second and third deal with the teacher's actions.

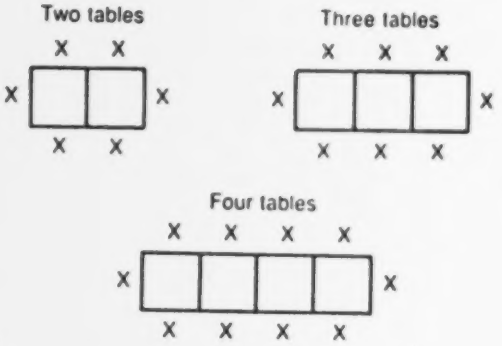
Instructional guideline 1: Select problems to develop a variety of problem-solving strategies

One- and multiple-step story problems are traditionally included in school mathematics programs. These problems are solved by choosing one or more operations—addition, subtraction, multiplication, or division. Although one- and multiple-step problems should continue to play a role in school mathematics programs, problem-solving instruction should not be limited to them. A *process problem* is another type of problem that should be included. Process problems cannot be solved simply by choosing one or more operations. Rather, they are solved by using one or more strategies, including *guessing and checking, drawing a picture, making an organized list, making a table, looking for a pattern, solving a simpler problem,* and *working backward.* An example from the intermediate grades is shown in figure 2.

Fig. 2

Sample process problem:
Tony's Restaurant has 30 small square tables to be used for a banquet. Each table can seat only one person on each side. If the tables are pushed together to make one long table, how many people can sit at the table?

Solution:
Simplify the problem.
Draw a picture.
Make a table.
Look for a pattern.



Number of tables	Number of people
2	6
3	8
4	10
5	12
6	14
•	•
•	•
•	•
•	•
30	62

Pattern: 2 times the number of tables ---plus 2

2 × 30 = 60, 60 + 2 = 62

62 people can sit at the table.

Experience with the variety of strategies used to solve process problems promotes the improvement of many of the processes involved in mathematical thinking. For example, experience in looking for patterns helps students develop their ability to make generalizations on the basis of examples; experience in drawing pictures improves students' ability to make mental representations of mathematical situations; and experience in guessing and checking improves students' ability to formulate hypotheses. Because many process problems can be solved using different combinations of solution strategies, they help dispel the beliefs that there is only one way or only one best way to think mathematically. Process problems also illustrate the value of organizing information and help students develop techniques for doing so (e.g., how to make an organized list).

Mathematical thinking often requires the coordination of several thinking processes. Teaching students techniques for understanding and solving process problems and techniques for evaluating their work, together with giving practice in solving process problems, can develop students' ability to coordinate thinking processes. Finally, because the time needed to solve process problems is usually greater than that required to solve other types of problems, experience with process problems illustrates for students that mathematical thinking is seldom instantaneous.

At the end of this article, several sources of process problems are given.

Instructional guideline 2: Use a process-oriented instructional approach

The instructional approach used for delivering a problem-solving program should focus "on having students dis-

cuss, think about, and generally become more aware of the various processes they use to solve problems" (Lester 1983, 239). The following process-oriented teaching actions can be used to guide students' work in solving a problem (Charles and Lester 1982):

1. Ask students questions to help them understand a problem before they try solving it.
2. Ask students to suggest possible solution strategies for a problem before they actually begin work on it.
3. If necessary, provide specific hints to help students overcome blocks in solving problems.
4. Ask students questions that require them to monitor and evaluate their thinking.
5. Have students explain their solutions and name the strategies they used to find the solution.
6. Compare and contrast other problems with the one solved.
7. Have students solve extensions of the original problem.

Such process-oriented teaching actions can develop specific thinking processes. For example, actions 2 and 6 can develop students' ability to relate new learning tasks to ones previously encountered. Action 3 can develop students' ability to select appropriate thinking processes. Action 4 can develop students' ability to regulate their thinking, and action 5 (i.e., proving that a solution is correct) can develop students' deductive thinking processes.

Instructional guideline 3: Establish a positive classroom atmosphere

Mathematical thinking seldom involves a simple, direct line of reasoning. Usually one has to play around with some ideas, make conjectures, play hunches, and generate sensible plans. If problem-solving experiences are to promote the development of mathematical thinking, they need to be presented in such a way that students are keenly aware of the freedom and desirability of exploring any ideas for solving problems. Also, concepts and skills from computational strands, estimation, mental arithmetic, geometry, and all other basic-skill areas can be used in a problem-solving program. When students have opportunities to use previously acquired knowledge in interesting ways, their attitudes toward mathematics and mathematical thinking improve. Unless students are willing to engage in mathematical thinking, they cannot develop their thinking processes. Problem solving can develop this willingness. The following are some additional guidelines for establishing a positive classroom environment for problem solving:

- Recognize and reinforce willingness and perseverance.
- Reward risk takers.
- Encourage students to play hunches.
- Accept unusual solutions.
- Praise students for getting correct solutions but emphasize the selection and use of problem-solving strategies.
- Emphasize persistence rather than speed.

Each of these instructional actions for promoting a positive classroom environment for problem solving contributes to the development of helpful attitudes and beliefs related to mathematical thinking. For example, accepting unusual solutions illustrates for students that mathematics problems can be solved in more than one way. Encouraging students to play hunches helps them realize that mathematical thinking does not always proceed along a path that everyone knows except them! And emphasizing the selection and use of problem-solving strategies affects students' beliefs about their ability to do mathematics.

Conclusion

Problem solving is the process of (1) understanding a problem, (2) selecting or collecting data needed to solve it, (3) choosing and implementing one or more solution strategies, (4) answering the problem, and (5) evaluating the reasonableness of the answer. Because it is a complex process, problem-solving experiences designed and delivered according to the guidelines presented in this paper have the potential to elicit a higher level of mathematical thinking than many other learning tasks that students encounter in the study of mathematics. In most problem-solving experiences, various thinking processes are needed, and the use of these processes is significantly influenced by students' ability to access relevant knowledge, to control debilitating attitudes and beliefs, and to draw on helpful attitudes and beliefs. Indeed, the ultimate aim of problem-solving instruction is the development of mathematical thinking.

Sources of Process Problems

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How to Teach Problem Solving

By Marilyn Burns

Our charge as elementary teachers has been made clear—problem solving is the number one focus for school mathematics of the 1980s. And that's exactly as it should be. It would be hard for anyone to deny that the major responsibility of schools is to help children become effective problem solvers, preparing them to face the myriad of day-to-day problems of living they'll face as adults. But it's also hard for teachers to relate that responsibility to what happens in the mathematics curriculum on a daily basis. This article aims to deal with just that issue—how to incorporate problem solving as an integral part of the mathematics program in the classroom.

The Typical Classroom Situation

First, a look at what the reality is in most elementary classrooms. The major thrust of mathematics in the elementary school is arithmetic, and this has been supported heavily by the back-to-the-basics movement. Children spend the bulk of their mathematics time in school practicing arithmetic skills, mostly with paper-and-pencil drill. (Can you estimate how much of your math-time children spend in this way?) NCTM launched the 1980s with the statement that problem solving is the real basic, that mathematics must deal with more than computational ability, that computation done in isolation is certainly not problem solving, and that conventional word problems are a limiting way to approach problem solving. Yet standardized tests, textbooks, and curriculum guides emphasize arithmetic; and even parents, school board members, and children see "doing arithmetic" as the major part of mathematics for the elementary school. What is the elementary teacher to do, especially the teacher who isn't comfortable enough with the scope of mathematical concepts that develop problem-solving abilities in chidren?

There is much that teachers can do in the classroom to foster problem solving, but this action needs to come from careful thought about children, about mathematics, and about school. First of all, the emphasis on computational skills needs to be examined in a realistic perspective. Making proficiency at doing arithmetic with paper and pencil the major concern of elementary mathematics is absurd. As the priority, it makes no sense in terms of our responsibility of preparing children for adult life.

A Perspective on Arithmetic

When do you use arithmetic in your daily life, outside of classroom responsibilities that stem from teaching computation? When teachers attending workshops are asked to answer this question, their responses usually include the following situations: when I balance my checkbook; when I am in the supermarket keeping note of how much I am spending; when I need to know how much wallpaper, or carpet, or floor covering I need; when I am figuring the tip to give in a restaurant; when I want to know what mileage I'm getting on my car; and when deciding how long I need to bake a roast or turkey. When asked to list all the different methods for doing arithmetic, these same respondents give answers that usually fall in these categories: with a calculator, with paper and pencil, mentally, using some manipulative. Finally, when these teachers review their first responses and note the method they usually use to do the arithmetic in each of the instances, the answers again show a consistency. Using a calculator or doing the arithmetic mentally are the two most consistently used methods; paper and pencil are not usually resorted to. Why then should children be spending the *major* portion of their mathematical time in school practicing what will be of little use to them as adults? (You might try this listing experience with the parents of your students to help them look at the arithmetic issue with a new perspective.)

This does not mean that children do not need to learn arithmetic. Being able to do arithmetic mentally demands mastery of basic facts coupled with the ability to estimate. Using a calculator successfully requires understanding of the arithmetic processes called for in a variety of situations. Practicing paper-and-pencil arithmetic in isolated worksheet situations, however, does nothing to ensure that our children will learn to apply computational skills when needed. That's

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clear to teachers whenever word problems are assigned, and it has been corroborated by research findings. There is no place in our schools for learning skills without careful development of number understanding stemming from situations that need those skills. Can your students tell you if it's cheaper to buy things two for a nickel or three for a dime? Can they give you a fraction that is larger than 1/3 yet smaller than 1/2? Can they tell you what they would do when halving a recipe that calls for 3/4 of a cup of sugar? Learning to compute with pencil and paper does not guarantee success with these kinds of problems that demand more than just computational facility.

What are the problem-solving skills that we need to be teaching? A look at problems as they appear in real life helps in considering that question. In real-life problems, you are rarely given all the information you need in one tidy package; you usually have to collect the data, and often from a variety of sources. Only rarely is there only one possible method or only one plausible solution that emerges from real-life problems; usually you choose one from several viable possibilities. You don't always know for sure if the solution you choose is the "right" or "best" one; it may be only later that you can evaluate your choice. Sometimes you never find out for sure; life has no answer book. When solving problems in real life, you call upon skills you've developed in other situations. Some problems demand skills you haven't learned or practiced very much.

In contrast, consider the kinds of mathematics problems that are typically presented to students in elementary-school classrooms. How much of the class time is being spent on arithmetic exercises or arithmetic word problems in which students have to find the one right answer, usually in one way, with the teacher giving the feedback as to the correctness of the solution? How much of the class time is being devoted to mathematical explorations in which many solutions are possible, where the emphasis is put on the process of solving problems, and students are given the opportunities to decide if their solutions make sense? And which of these two approaches will better prepare children for dealing with the real world?

Organizing the Classroom for Problem Solving

There are two components to the issue of problem solving for the classroom teacher. One is the practical side: What do I do with the children to promote problem solving? The other is the theoretical side: What is my rationale for the choices I am making in the classroom? There is no separating these two aspects of responsible and effective classroom practice. Theory without practice is useless; practical application without a theoretical base is irresponsible. It's a blend of the two that teachers need to achieve.

What's clear is this: If you want to teach your students to be problem solvers, then they must have opportunities to solve problems, as a regular and ongoing part of their daily classroom experience. Developing problem-solving abilities is a long range goal, not a lesson objective. That means children need a steady flow of problems to solve. They need to be in a classroom environment where problem solving can best occur and to value the process of solving problems above the attainment of answers. They need models of problem solvers to emulate who are active learners; seekers who are willing to plunge into new situations, not always knowing the answer or what the outcome will be.

Your role in the classroom is key. You need to emphasize persistence, not speed, in tackling problems. It's important that you focus on the thinking involved, not just the answer. (So much of our curriculum is answer-oriented; here's the perfect opportunity to make the curriculum process-oriented.) You must be willing to learn along with your students, and to share your own striving with your students when some problem is difficult, so they can see you struggle. Most importantly, children need to see you, their teacher, as a learner; willing to explore and even being wrong at times. Not knowing does not need to be perceived as failing; it can be thought of as not having found out as yet.

Make problem solving obvious and visible to your students. Talk about problem solving with them. Let them know your viewpoint. Let them hear about how you value problem solving. Children need to get clear messages about what is important in their schooling.

Support cooperative problem solving in your classroom. So much of a child's school achievement is individual, or seen in the light of competition with others. Here's the chance in the classroom where more heads are truly better than one. Since the perspective in problem solving is on the processes children go through as well as on the answers, there is no need for children to have to work individually as a general rule. Your goal is not to assess their individual abilities to solve a variety of problems, or it shouldn't be. Developing those individual abilities can best happen when students experience problem solving in a positive way, in situations where success is imminently more possible than failure, and where they have all the support possible for learning about themselves as problem solvers.

Pedagogical Background

Piaget has offered powerful and useful notions about the necessary ingredients for children's intellectual development. There are four factors, he says, that contribute to optimal growth: maturation, physical experience, social interaction, and equilibration. Maturation is one that you can't do much about as a teacher; the older children are, the more likely they are to have mental structures that act in coordinated ways. Elementary-school children are not fully matured and you as a teacher have to work with what you get.

The other three factors, however, offer specific guidance for classroom decisions. To support the need for physical experiences, you can provide problems for your students that involve physical objects; problems in

which the materials themselves offer feedback as to the effectiveness of their thinking and solutions. The more experiences your students have with physical objects in the environment, the more likely that related understandings will develop. This holds true throughout the elementary school years, not merely in the primary grades. For social interaction, the more opportunities your students have to interact with others—with you and with their peers—the more opportunity they will have to gain perspective on their own viewpoints and approach objectivity. Equilibration is the one factor that coordinates the other three. It involves a continuous interaction between the child's mind and reality. Equilibration is the process children go through to balance what's going on inside their minds (their mental structures) with new experiences from outside.

Along with Piaget's ideas, examination of recent findings about the brain can help in formulating classroom procedures. There is no set teaching sequence that guarantees to make children effective problem solvers, but the brain is where that learning occurs. Strategies are useful tools for tackling problems, but developing those strategies in children is not dependent on any hierarchical developmental presentation. That's not how the brain works.

The brain is an active organ, seeking to glean understanding from the information it receives, often from complicated and chaotic information. The brain is not a sponge-like, passive organ. Nor does it learn in predetermined sequences as stimulus-response psychology suggests. As explained by Leslie Hart in his book, *How the Brain Works*, what the brain does is extract meaning out of confusion by looking for and recognizing patterns in the confusion. This process does not have to be taught; this is what the brain does, naturally and aggressively. To extract the essential patterns, which lead to concept development, high input of experience is essential. Young children don't learn to distinguish between chairs, stools, couches, and benches by seeing one example of each. They see many examples, in many situations, being used in different ways, and from the totality of their experiences they learn to discriminate. What children in our classrooms need is a wide variety of problem-solving experiences from which they can develop, within their own minds, those problem-solving strategies that lead to success. Teaching problem solving is not a logical, well-sequenced process. The key is high input, in a supportive, stress-free environment. This is what Hart has termed "brain-compatible teaching."

From Theory to Practice

Putting this all together seems to prescribe a particular kind of classroom setting: one where children are working together, encouraged to share their ideas with each other and with you, and dealing with concrete materials from which conceptual understandings can be created. Students need to face problems that are new and challenging, that force them to evaluate and modify their own mental structures as new information becomes available.

Our responsibility as teachers is to implement an active and ongoing component of problem solving into students' mathematics programs. Nothing should stop this. Start small, if you must; but start. Mull over the ideas presented here and in the other articles; consider your particular students and find a way to institute problem solving with enthusiasm along with clear information for your students.

The following describes how I introduced problem solving to a class of sixth graders: "For one period a week, we are going to be doing problem solving. What this means is that I will present some problems for you to solve. The problems will be mathematical problems, but they won't be arithmetic problems. Usually in arithmetic problems, there is one right answer, and your job is to get to that answer. The problems we'll be working on during this period don't always have to do with arithmetic, and they don't always have getting a right answer as the goal. You'll see what I mean when we get started. And with this new kind of work, you won't be working alone on problems. I'm going to have you work in small groups, where the task will be for the entire group to solve the problem. You won't be alone with a problem; you and your group will be in it together."

All that was very abstract for them, so I plunged into a problem, one that was new for them and, I hoped, interesting. I chose a problem that was not dependent on prerequisite skills, that provided for cooperative interaction within the groups, and that allowed for flexibility as to how they approached the problem. I gave them squares of construction paper, measuring about four centimeters on a side. Their group problem was to find all the possible ways to arrange five squares so that each square completely touched the edge of another. There's nothing new about this problem; it's the pentomino problem that was presented years ago in the *Arithmetic Teacher* ("Geometry of a Flattened Box" by Marion Walter, May, 1969). We discussed that flips and rotations of the same shape were not to be considered different. I did not tell them that there were 12 possible shapes. I wanted them to explore the issue of how do you decide when you've found them all, along with the spatial task of finding possible shapes. The children worked in their groups, with concrete materials. Afterwards, they reported back to the entire class for the purpose of hearing other groups' methods and thoughts.

In a second session, groups were to find and record all the ways to write the numbers from 1 to 25 as the sum of consecutives; e.g., $9 = 4 + 5$ and $9 = 2 + 3 + 4$. I requested one record from each group that also included summary statements about what they noticed. I told them that it was not possible to write some of the numbers as the sum of consecutives, and to see if they could figure out the pattern of those numbers. They posted their work, allowing comparison for the purposes of discussing differences in recording techniques and exploring their generalizations. The emphasis was not on "right" or "wrong," but on "how".

It is not only the older children who need problem solving. Asking a group of two, three, or four children to share a banana or an apple or a box of raisins is a task that is appropriate for even young children. Once the groups have completed the task to their satisfaction, it is valuable for them to share with the class how they approached the problem and how they evaluated whether they had succeeded. (This problem is interesting for adults as well, especially with three in a group sharing one apple or banana.)

Giving a group of children the problem of covering a supermarket carton for classroom storage is appropriate and practical as well. Having a group guess how many and then count the number of beans (or some other objects) in a jar supports problem solving as well as estimation.

Final Thoughts

Providing problem-solving experiences in your classroom means finding suitable problems to present to your students. You need to introduce the problems, allow the children the time they need to explore, and then summarize their results with them, drawing on all of their ideas to lead them to appropriate and realistic conclusions. Your focus needs to be on stretching their thinking. The point of this article is not that this emphasis is to replace your concerns about arithmetic skills, where your focus may be on accurate results, but rather, that both aspects of your mathematics program are essential, for all children, at all grade levels.

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Reviewing and Discovering

(Continued from p. 30)

Story Problems: Addition Facts, Subtraction Facts, Mixed Addition and Subtraction Facts, Two Digit Addition and Subtraction. *David C. Yeager. 1981. $4.95 ea., or $19 complete series. Complete series in a binder, $22. Creative Publications, P.O. Box 10328, Palo Alto, CA 94303.*

This is a series of four booklets on (1) addition facts, (2) subtraction facts, (3) mixed addition and subtraction facts, and (4) two-digit addition and subtraction. The booklets also come in a loose-leaf binder form for easy reproduction, although the booklets have tear-out pages. Each page of the booklets is a one-page cartoon presented in 5 to 6 frames, followed by five questions. The cartoon dialogue carries all the information needed for the computation required to answer the questions.

Reading comprehension is considered to be an integral part of the problems, and the Dolch word list for grades 1–3 was used to provide consistent vocabulary. All non-Dolch words are listed for each lesson, at the end of a booklet. Some teachers might question whether the Dolch list is too dated to be useful in a context such as this.

No grade levels are indicated in the materials, but a commercial catalog lists this set as appropriate for grades 3–6. There is some question in my mind as to whether the problems are mathematically appropriate for the upper grades, since they deal primarily with addition facts through 10 + 10 and related subtraction facts, with only the last booklet including any two-digit addition and subtraction. Although the arithmetic is accessible to first and second graders, the reading level is not.

Children will find most of these stories entertaining. There are times when the story line is a little obscure. The drawings are somewhat busy and at times deceiving (for example, the dialogue in one frame refers to three fishing lines when four are pictured). The drawings are obviously not meant to assist the student in an organized manner. The problems presented by the dialogue are actually the traditional type of word problem. The cartoon format, however, provides a motivational aspect not found in most textbook word problems.

The major problem-solving skills required in the first two booklets consist of searching the frames for the one with appropriate information, then doing the computation required to answer a question. The last two booklets call for decision making, about the type of operation to be used and about which information is relevant.—*Judith Threadgill-Sowder, Northern Illinois University, DeKalb, Illinois*

Classical Problems for All Ages

By **Rosalie Jensen** *and* **David R. O'Neil,** *Georgia State University, Atlanta, Georgia*

Most children in the later primary grades and throughout the middle grades are functioning at the stage of concrete operations. When they are faced with genuine problem-solving situations they need concrete and pictorial aids, as well as guidance from adults in organizing information and choosing strategies.

Many teachers present real-life problems to help children develop problem-solving skills. Another source of interesting problems for maturing students is the large body of problems that can be found in old textbooks and books of mathematical puzzles and challenges available in libraries and book stores. Some of these old problems have entertained thinkers over the years and many do not require complicated computations or advanced mathematical concepts. By modifying the situations in the individual problems, a teacher can adapt problems to different grade levels and abilities and develop problem-solving opportunities for students that focus on strategies of solution rather than on the performance of complex arithmetic operations. Three such examples are included in this article.

Through the professional literature researchers and teachers have shared recommendations for helping students find solutions to problems. Some of the specific approaches appropriate for elementary- and middle-school students will be illustrated in the solutions to three problems of the classic type that are described in this article.

On a general level, three things should be made clear to students:

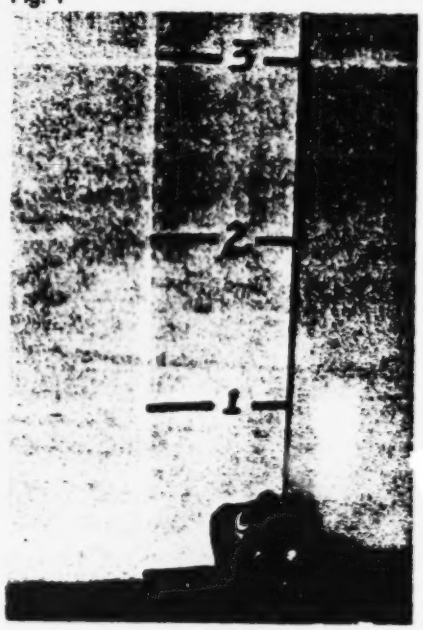
1. Several strategies may need to be tried before a suitable one is found.

2. There may be more than one solution to a given problem.

3. Sometimes one must back away from a problem for a while in order to get a fresh look before proceeding.

In a classroom setting different strategies for solving a particular problem may be undertaken by individual students or may be pursued through large group discussion. Small groups of students might attempt to solve the problem using various strategies and then present their solutions to the whole group. The approaches illustrated here can be used effectively with any of these organizational patterns.

Fig. 1



First Problem

Getting out of a hole

Kermit the frog fell in a hole that was 10 meters deep while he was taking a walk in the woods. Each day he could only jump 3 meters up the side of the hole. Each night he slid down 2 meters. How many days did it take Kermit to get out of the hole?

Understanding the problem

Sometimes it is necessary to understand why a particlar situation constitutes a problem. Encourage children to talk about the problem. You may need to pose questions to initiate their discussion or to provide selective hints. Depending on the class and the course of the children's discussion, questions like the following might be asked: Is the number of days the same as the number of jumps? If we know the number of jumps do we know the answer to the problem? Do we need to be able to measure in meters to find the answer? Would the answer be different if Kermit did not slide down each night?

Collecting information

One way to help children process the information in the statement of the problem is to guide them in selecting the things that are important. Sometimes it is helpful to decide what we are looking for before we consider what we are given. In this problem, will the answer be a number of frogs? a number of meters? a number of days? a number of jumps? Obviously

we are looking for a number of days. Now what do we know that we can use in finding the number of days?

Ask pupils to select items from the following list that they need to find the solution:

The frog's name is Kermit.

Kermit was taking a walk when he fell in the hole.

The hole was 10 meters deep.

Kermit jumped up 3 meters each day.

He slid down 2 meters each day.

The important information might be summarized on the board into categories of "What we need to find" and "What we already know." At this point in the development some children may wish to guess the answer. This is all to the good since children who make guesses will be especially motivated to find the "real" solution.

Divising and implementing a plan

The nature of a problem dictates the approaches that will lead to a solution. For this problem we will consider three different methods, but you or your students may think of other ways to solve it.

Using manipulative aids. Place ten evenly spaced marks on the wall or chalkboard to represent the 10 meters on the side of the hole. Let a child demonstrate the solution by using a toy frog (fig. 1). Have several children check the solution by manipulating the frog.

Using a diagram. Give each child a drawing that looks like figure 2. The solution can be found by drawing arrows as in figure 3. Children who have solved addition and subtraction problems by using a number line will be familiar with this type of soluton.

Making a table. Collecting and organizing data is an important problem-solving technique. Figure 4 shows a table that was constructed using the information given. The solution is circled.

Extending the problem

Once the problem has been solved, you might ask the children to write a similar problem by changing the information in the problem. Then let the children solve some of the problems that have been written. Can all the problems be solved by the same strategies?

Fig. 2

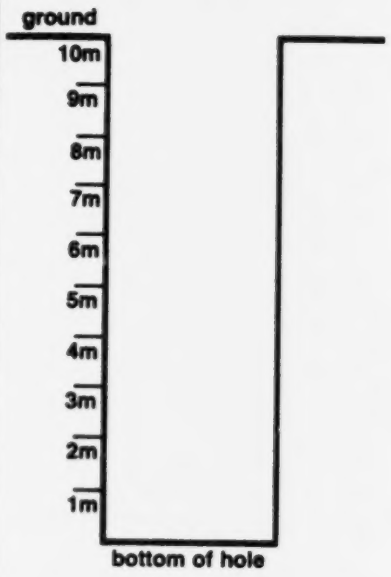


Fig. 3

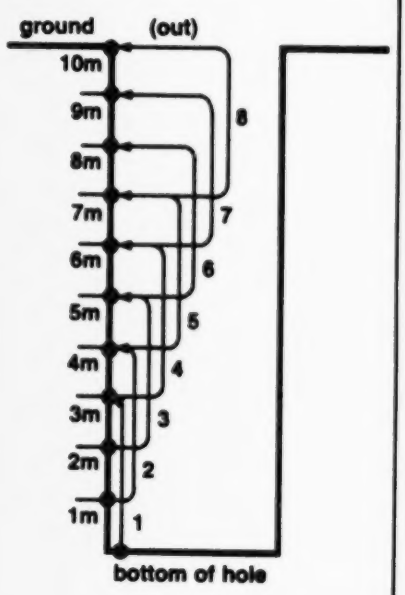


Fig. 4

Day	Hops up to	Slides down to
1	3m	1m
2	4m	2m
3	5m	3m
4	6m	4m
5	7m	5m
6	8m	6m
7	9m	7m
(8)	10m (out)	

Second Problem

Measuring water

Ms. Green, the art teacher, needs 2 liters of water to make a clay mixture for her students. She has two unmarked pails, a 3-liter pail and a 4-liter pail. Using only these pails, how can she get 2 liters of water?

Understanding the problem

Encourage children's comments and questions about the problem. Someone may suggest, "She can measure two liters." The pails are unmarked, however, so she can't measure anything but 3 liters or 4 liters.

Someone may propose, "Fill the four-liter pail half full." Interestingly, if the pails are cylindrical this can be done by tipping the pail until the water reaches the rim. A student who is clever enough to determine how to fill a 4-liter cylinder half full should be given credit for this answer. Meanwhile, assume that the pails are not cylindrical but are shaped more like paper cups.

Collecting information

Help students pinpoint the important parts of the problem statement by looking at all the information and se-

Fig. 5

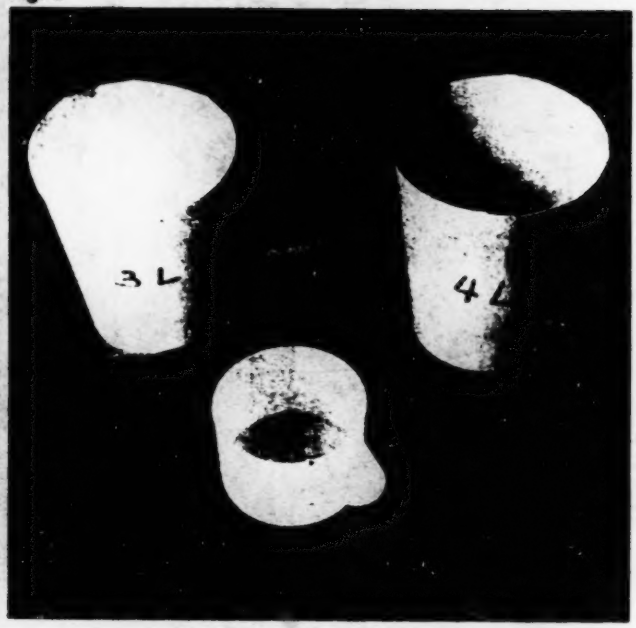


Fig. 6

To do	Get
Fill 3-L pail	3L
Fill 4-L pail	4L
Fill 3-L and 4-L pail	7L
Fill 4-L pail and pour into 3-L pail	1L
Fill 3-L pail, pour into 4-L pail. Fill 3-L pail again	6L
Start with 6-L. Pour water from 3-L pail to fill 4-L pail.	2L in 3-L pail

lecting crucial items. Display this list of items on an overhead projector, write them on the chalkboard, or include them on a hand-out page.

Having identified the important considerations, lead students into answering the following questions:

What is needed? (Two liters of water.)

What is given? (Two unmarked pails that hold 3 liters and 4 liters respectively.)

What relationships hold between the thing that is needed and the things that are given? (No obvious arithmetic relation. For example, $4 - 3 \neq 2$, $3 + 4 \neq 2$, and so forth.)

Devising and implementing a plan

This problem is more difficult to solve than the problem involving the frog in the hole. The approaches outlined here might take place over several periods of instruction. You should encourage promising ideas and interact with students, giving hints without telling them how to solve the problem.

Using manipulatives. Suppose that some students decide to use manipulatives and to record results. To represent the pails in an appropriate manner, they might measure 3 scoopfuls of water in one paper cup and 4 scoopfuls in another, marking the level of water in each cup. By cutting the cups at the marks and by labeling them the students will have reasonable models for the pails (fig. 5).

The results of experimentation can be recorded as a sequence of steps:

To get 4 liters, fill 4-liter pail.

To get 3 liters, fill 3-liter pail.

To get 7 liters, fill 3-liter pail and 4-liter pail.

To get 1 liter, fill 4-liter pail and fill 3-liter pail from 4-liter pail. Have 1 liter in 4-liter pail.

When students reach 1 liter they may wonder if they can pour the liter of water in a third container and add another liter of water to it. You may either accept this solution or offer a suggestion for proceeding toward a solution based on using only the two available containers. Eventually the students will discover that they can transfer the 1 liter from the 4 liter pail to the 3 liter pail, fill the 4 liter pail, and pour as much water as possible from the 4 liter into the 3 liter pail. Having done this, they will have 2 liters of water in the 4 liter pail.

Using a table. Figure 6 shows a table constructed by imagining all possible actions and recording the results of the actions. If students reach 6 liters and have difficulty in continuing, you can suggest that they review what has already been done to determine if the actions can be taken another step. Perhaps they will discover the solution recorded in the last row of the table. Notice that this is a different solution from the one described previously.

Considering simpler problems. A third possible strategy is to consider simpler, related problems. For example, replace the 2 liters needed to solve this problem by other quantities. It would be easy to obtain either 3 liters of water or 4 liters of water by filling a single pail. Since $3 + 4 = 7$, we can obtain 7 liters by filling both pails. Students should be able to find similar interpretations for the combinations $4 - 3 = 1$ and $3 + 3 = 6$. Thus, the numbers 1, 3, 4, 6, and 7 can each be substituted for 2 in order to create a simpler problem. Suggest that students find combinations of these numbers that have 2 as the result. Possibilities are $3 - 1 = 2$, $6 - 4 = 2$, and $1 + 1 = 2$. Using the combination $6 - 4 = 2$, students can reach the same solution as the one illustrated in the table in figure 6 by a different route.

Extending the problem

While they are solving Ms. Green's problem, students will be finding ways of obtaining 1, 2, 3, 4, 6, and 7 liters of water using only a 3-liter pail and a 4-liter pail. Is it possible to

ɔtain 5 liters of water using these ıils? If we changed the capacities of .e pails to 4 liters and 5 liters, can we ɔtain every whole-number amount p to 9 liters? Students can solve ıese problems and write their own roblems using different combinations f numbers. Do some of the problems ave several solutions? How many ɔlutions can we find? Are some of ıe problems impossible to solve? Answering questions like these can help tudents become true problem solvrs.

Third Problem

Distributing weights

After a forest fire in a national park, the park rangers must relocate eight animals whose homes were destroyed. Each van used for this job can carry no more than 100 kilograms. The weights of the animals in kilograms are 45, 38, 32, 26, 21, 16, 9, and 2. Help the forest rangers by finding at least one way to put all the animals in the two vans available.

Understanding the problem

Sometimes, to help children understand the nature of the problem, you may find it useful to give them hints by asking questions like these: Can all the animals be put in one van? Why? Does the statement of the problem tell us how many animals to put in each van? Must all the animals be placed in two vans? Will the answer to the problem be stated as a number of animals? What kind of answer do we need? What mathematical operation(s) will we use in solving the problem?

Collecting information

The initial discussion of the problem will suggest that to simplify the organization of the information in this problem you think of the given weights as eight numbers that are to be combined in a special way. The eight numbers are to be separated into two sets and the sum of the numbers in each of the two sets is not more than 100. A number can be used only once and all numbers must be used.

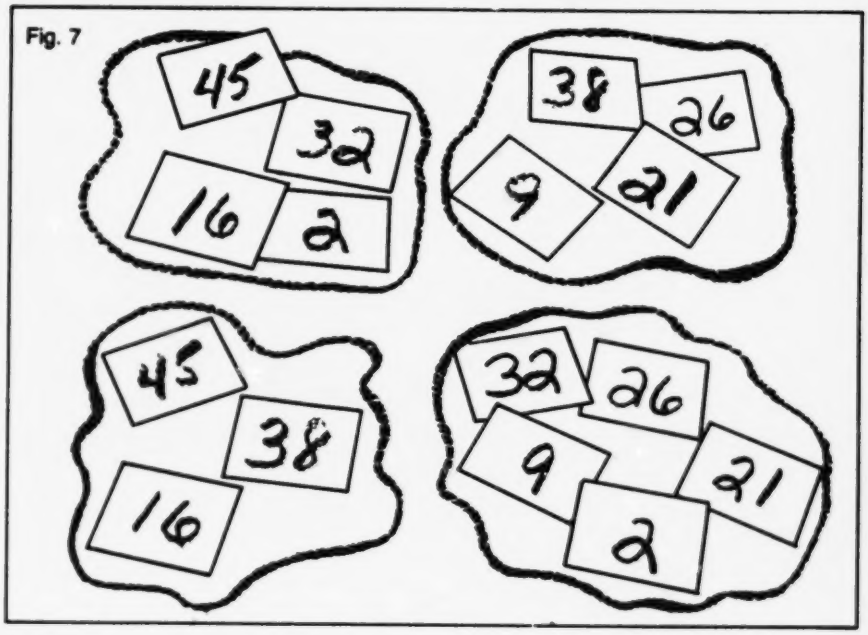


Fig. 7

Devising and implementing a plan

Emphasize the fact that different solutions exist for this problem. If the class finds a solution through large-group discussion, individuals may wish to find additional solutions.

All of the methods described here require a trial-and-error strategy initially. At some point in the process students should begin to realize that it is more efficient to try certain combinations of numbers.

Using a manipulative model. For this approach you will need a pan balance, gram masses (weights), and modeling clay. Have pupils make balls of clay that have masses of 45 g, 38 g, 32 g, 26 g, 21 g, 16 g, 9 g, and 2 g by placing the appropriate gram masses in one pan and the clay in the other pan. Mark each completed ball with its mass. Separate the balls of clay into two sets so that the weight of each set is not more than 100 grams.

Pupils may decide to sort the balls of clay by placing them in two piles so that the total volume in each appears about the same. (This is a good strategy since all the balls are made of the same material.) If a solution to the problem is found, record the solution. Have the children continue shifting balls from one pile to the other in order to find additional solutions.

Using number cards. One way to reduce the frustration of record keeping for this problem is by writing each of the given numbers on a small card or piece of paper. The cards can then be sorted into two piles (fig. 7). The children may discover for themselves that they need to put some of the large numbers in each set.

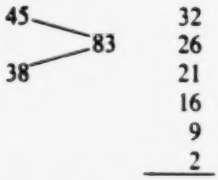
As children make their combinations of numbers, have them add the numbers in each set using pencil and paper or a hand-held calculator. Suppose that one trial looks like this:

45	32
38	26
16	21
2	9
101	88

One of the sums is greater than 100. By exchanging the 38 and the 32 in the two sets, a solution can be obtained. Are there any other exchanges that will yield a solution?

Using a hand-held calculator. Even without manipulating cards, it is possible to arrive at a solution efficiently with a calculator. Begin with the first number in the set and add successive numbers until a sum greater than 100 is obtained. For example, 45 + 38 +

32 = 115. Too much! Write 45 and 38 in one column and the other numbers in another column.



Two actions are possible at this point:

1. Add different numbers in the second column to the sum of 45 and 38 until a solution is found, 83 + 16 = 99. Find the sum of the numbers that remain in the second column. Since 32 + 26 + 21 + 9 + 2 = 90, a solution has been found.

2. Subtract 83 from 100 to obtain 17. Find a number in the second column that is less than 17 but as close to 17 as possible. Add this number, 16, to 83 and add the numbers in the other column to determine if that sum is less than 100.

Because the numbers were arranged in order from largest to smallest both of these strategies will yield a solution in an efficient manner. Putting information in some kind of order is a good starting point for the solution of a problem.

Extending the problem

Ask students to suggest situations in real life that lead to problems like this one, which is called a bin-packing problem. Have them write story problems about farmers placing produce in baskets, about shipping clerks packing goods into cartons, and so forth. Pupils will receive practice in handling numbers of different sizes as they decide on maximum weights for the contents of cartons and as they choose weights for the items in such a way as to be sure of a solution.

Problems for Your Pupils

Three examples have been given of classical problems that can be adapted for presentation to elementary and middle grade students. They were selected from a large body of problems that have fascinated thinkers for centuries. In addition to written problems such as those selected here, people have been challenged to solve number-combination problems such as magic triangles and magic squares and geometric problems such as making specific configurations with tangram pieces.

When choosing problems for your own students, simply follow these steps:

1. Work out each problem ahead of time to gauge its difficulty.

2. Alter the problem if necessary by substituting data that will make computations simpler or will diminish the number of steps required for a solution.

3. Think of different strategies, such as those illustrated in this article, which can be applied successfully in finding solutions.

When presenting each problem follow a sequence of steps, such as the ones used here, which are appropriate for solving a large number of problems. Encourage students to try different strategies and accept diverse solutions which are suitable.

Problem solvers are not developed overnight. Have patience and remember that the end product of your work is the development of the most worthwhile intellectual endeavor, problem solving. ●

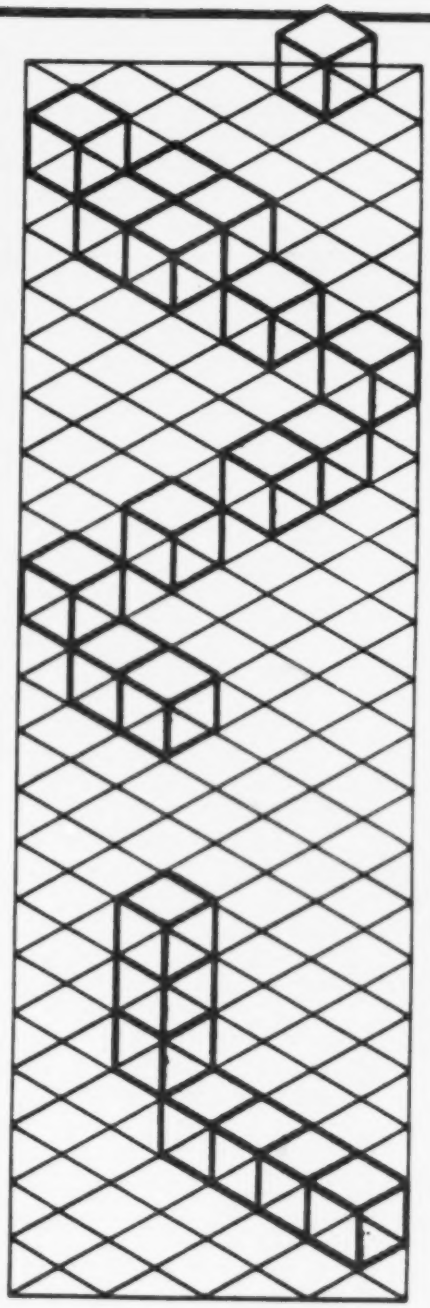
Problem Solving for the Primary Grades

By **James V. Bruni**

"How can my children possibly be expected to do *problem solving* when they are only *beginning* to learn to read and to do basic arithmetic computation?" This is often the lament of primary-school teachers when urged to include problem solving in their mathematics programs. Developing problem-solving skills is often equated with training children to solve arithmetic word problems, but problem solving has a much broader meaning. It is not only possible for young children to become problem solvers but also, primary mathematics instruction can be organized to help develop problem-solving skills.

Before suggesting how this can be done, let us consider what is meant by a *problem*. A real problem involves a question that cannot be immediately answered; it requires some effort in making appropriate use of previously learned concepts and skills for its solution. Young children are *natural* problem solvers. They have been successfully solving problems all their lives.

Think of the impressive *problems* children have solved before coming to school. On their own they learned how to get from one place to another (rolling over, crawling; and then walking, hopping, skipping and running) and how to communicate orally (by crying, making a wide assortment of different sounds for different purposes, and finally breaking the language code and learning to speak in a progressively more sophisticated manner). As you think about the myraid of obstacles they have overcome and problems they have solved, it becomes obvious that when children come to school they are already accomplished problem solvers.



The primary-grade teacher needs to help nurture problem-solving skills. It is extremely important for primary-school children to remain convinced that they *are* good at solving problems and to be involved in activities that help develop and refine their problem-solving skills in mathematics. All too often, children enter first grade with enthusiasm, anxious to try any mathematical problem and confident of their success, only to become "turned-off" by mathematics before entering the third grade, firmly convinced they cannot do it.

How can you nurture those important problem-solving skills that children have when they enter school? Here are six very practical ways a primary-grade teacher can help promote the development of problem-solving skills:

1. Ask more open-ended questions.
2. Encourage the discovery of patterns.
3. Help children develop spatial imagery and spatial thinking.
4. Focus on mathematical language development.

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5. Help children solve real-life problems.
6. Observe and reward the *different* ways children solve problems.

The remainder of this article offers specific examples of how these suggestions might be implemented.

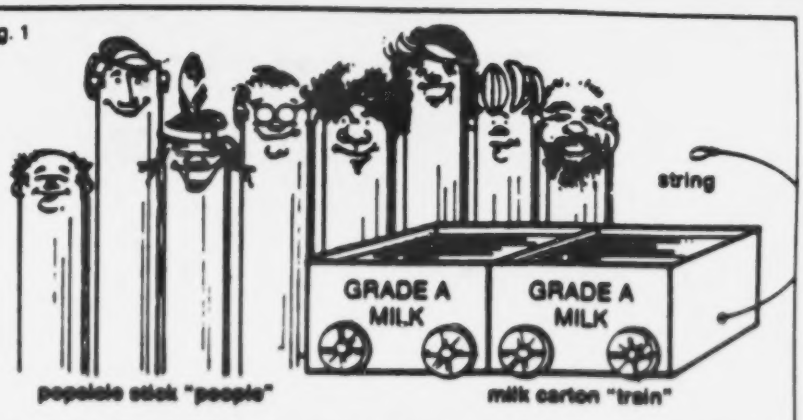
Asking Open-ended Questions

Activities such as those described in figure 1a for developing basic addition facts can be very effective in providing concrete, manipulative experiences that form a basis for understanding an abstraction such as 3 + 2 = 5. But even if a program is rich with concrete experiences, children can easily conclude that to do well in mathematics you not only have to figure out the one answer the teacher expects you to give, but also you must find the answer in the way the teacher expects you to find it. By asking some open-ended questions—questions that might evoke different responses or have more than one answer—you can help children see that there are usually many ways to solve a problem and that a problem can have several solutions (or none).

Figure 1b suggests how you can extend the activity described in figure 1a. You can use this same kind of open-ended activity to develop all the addition number facts. As a further extension, children can reinforce their discoveries by making number books for each number as described in figure 2. The pictures on each page can be gummed stickers, pictures cut from magazines, or children's drawings. These books represent a record of their different "solutions" to the problem.

Another example of a more open-ended activity involves collecting an assortment of empty containers from familiar products sold in the supermarket (e.g., cereal boxes, egg cartons, milk containers, and so on). You can put price tags on these containers, using numbers appropriate for your children, and display the collection on a table. Then ask questions about the containers as suggested in figure 3. Figure 4 suggests another activity that

Fig. 1



a. Put 3 stick people in the front of the train and 2 stick people in the back. How many stick people are in the train altogether?
b. Take 5 stick people for a ride in the train. How many will you put in the front? How many in the back?
Do it a different way.
How many different ways can you do it?

Fig. 2

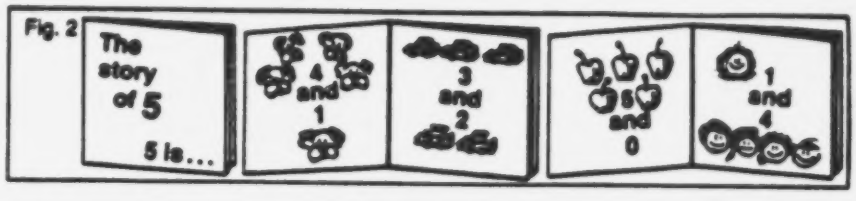
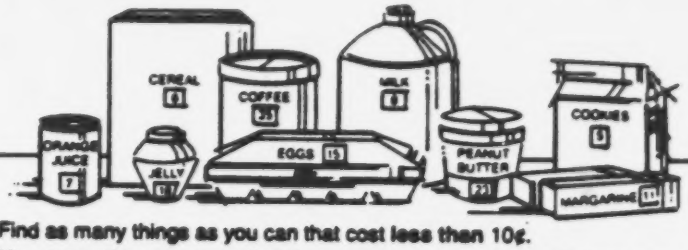


Fig. 3



(1) Find as many things as you can that cost less than 10¢.
(2) How many things can you find that cost more than 20¢? More than 50¢.
(3) Can you spend exactly 20¢? Which things could you buy?
(4) What is the largest number of things you could buy for less than 50¢ altogether? List them.

Fig. 4

Sorting Buttons

Materials:

Box of assorted buttons, 3 or more empty containers.

Put the buttons you think belong together into the same container.

How did you sort them?

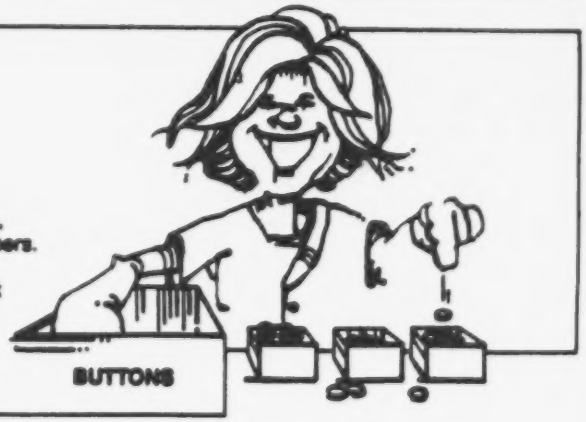
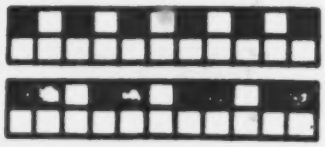
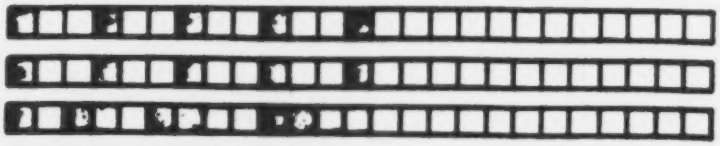


Fig. 5

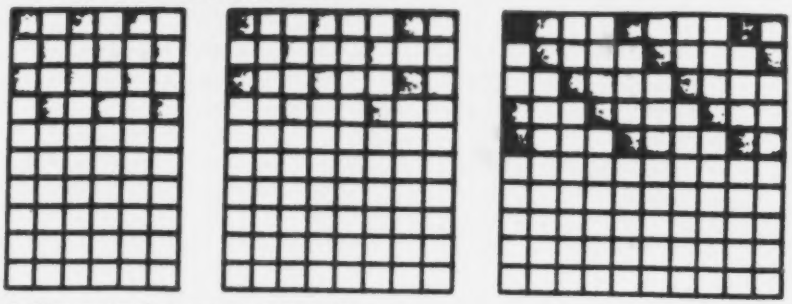
a. Can you copy these patterns?



b. Can you continue these patterns?

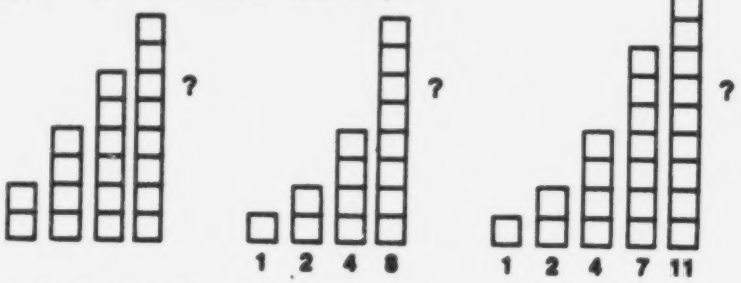


c. Can you continue these patterns?

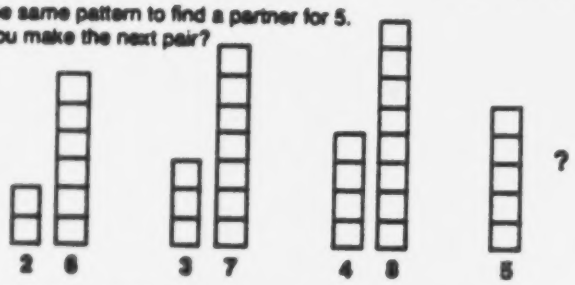


d. Can you make you own patterns?

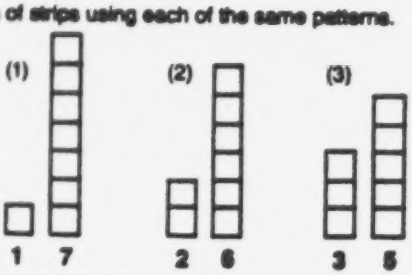
Fig. 6 a. What should be the next strip?



b. Use the same pattern to find a partner for 5. Can you make the next pair?



c. Find other pairs of strips using each of the same patterns.



can be made more open-ended through different questions.

Looking for Patterns

Recognizing patterns and determining relationships is at the core of thinking mathematically and solving problems. Young children enjoy opportunities to discover patterns and to create patterns. Beads on string, color cubes, or pegs on pegboards serve as a good introduction to patterning. In figure 5, children are asked to reproduce a pattern, or to continue a pattern in one or two directions, or to make a pattern of their own.

Using strips of squared paper as shown in figure 6, you can provide a transition from patterns involving squares to patterns involving numbers. The activities that use the strips of squares are only a few of the many possibilities.

Using Spatial Thinking

From their earliest years, children are involved in such spatial-thinking experiences as distinguishing one shape from another, recognizing specific shapes when they are in different positions, and transforming shapes. These kinds of spatial-thinking activities occur often during preschool years as children play with wooden blocks, put together jigsaw puzzles, and make cut-outs from paper. Such practical experiences with spatial relationships are helpful in building a foundation for developing the imagery or models for other mathematical relationships. Most mathematics programs use two-dimensional or three-dimensional models for developing number and fraction concepts, basic operations, and measurement concepts and skills. Yet regular attention to the development of spatial thinking is often ignored.

The collection of familiar containers shown in figure 3 can be used to help children explore the different shapes that form the faces of three-dimensional objects. Figure 7 suggests an activity in which children attempt to match containers with cards showing outlines of the faces. Some children may then be able to make these types of outline cards from other containers and add to the collection.

Wooden cubes can be used for a variety of spatial activities. One possibility is making models of cube structures by gluing cubes together in various ways (fig. 8) and asking children to reproduce the models with individual cubes. Drawings of models can then be used as an even more challenging activity (fig. 9).

By making wooden-cube structures of one, two, three, and so on cubes, you (and the children) can develop all sorts of challenging spatial activities in which specified sets of cubes are used to cover different regions (fig. 10). This activity can be extended to include more varied wooden-cube structures and to develop more complicated two- or three-dimensional models.

The idea of a tangram suggests another source of useful activities. By labeling the pieces as indicated in figure 11, children can use cardboard or oaktag pieces to explore how these pieces can be combined in different ways. Activities like these can be made very simple or very challenging.

Developing Mathematical Language

Results from the National Assessment of Educational Progress, as well as teacher observations, clearly indicate that many children have difficulty with word problems even when they can easily perform the necessary arithmetical computation. At the primary level it is essential that a great deal of emphasis be placed on developing the special use of language found in mathematics books. Children need many opportunities to use such terms as *more than, less than, as much as, most, least, equal to, as many as, altogether, add, subtract, difference, fewer than, greater than, twice as much as, half as much,* and so on. These terms should be used in as many different contexts as possible.

A useful way to develop an understanding of typical word problems is to have children act them out. Choose children to perform the action indicated by a problem. You can have several arithmetic sentences on the board

Fig. 7

Which supermarket container has faces that match the outlines on each card?

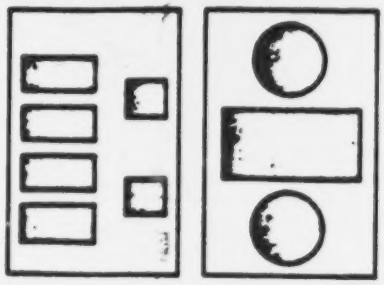


Fig. 8

Can you make buildings like these with cubes?

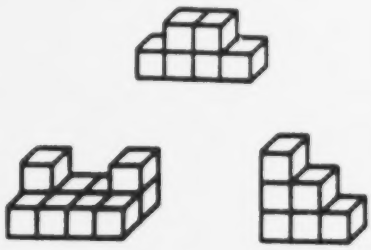


Fig. 9 Can you make buildings like those in the pictures using wooden cubes?

Fig. 10

Use the pieces indicated to cover each shape.

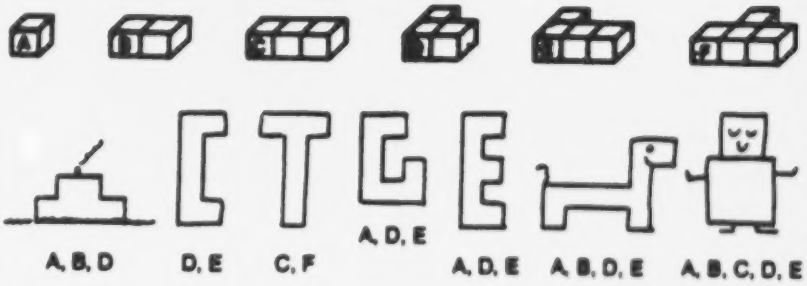


Fig. 11

Cover each shape using the pieces indicated.

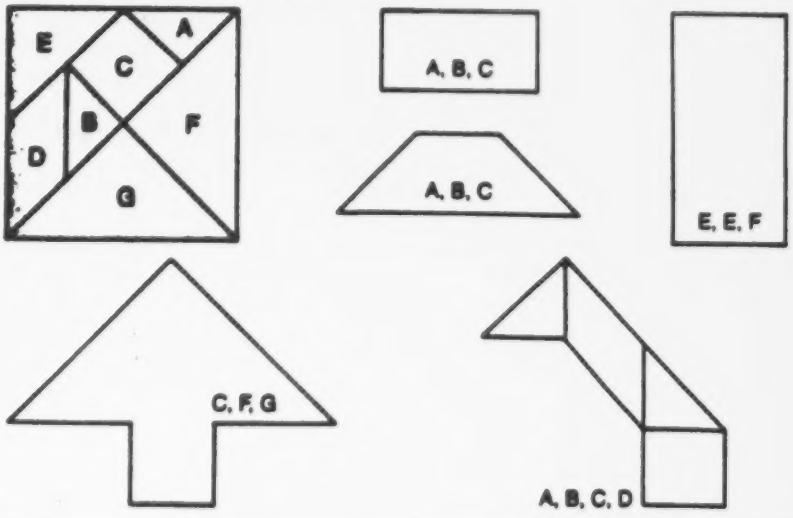
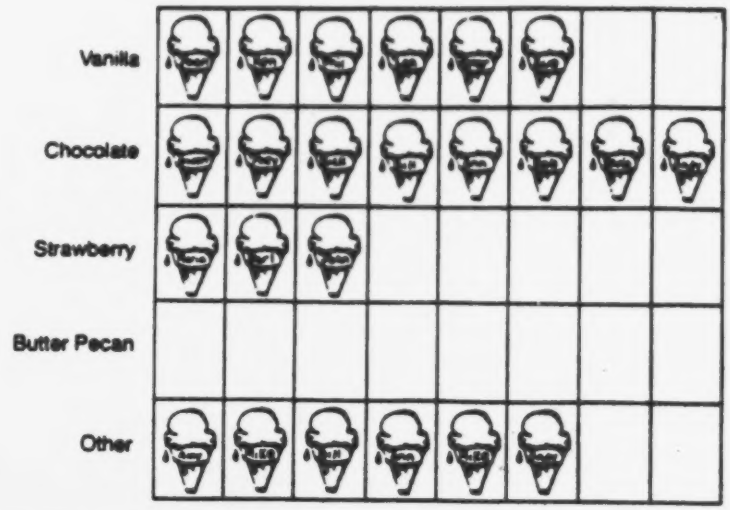


Fig. 12

What flavor of ice cream do you like best?



How many more chose chocolate than vanilla?
How many altogether choose vanilla or strawberry?
What does the graph tell us?

(1) Most children in the class chose chocolate.
(2) No one chose butter pecan.
(3) Fourteen children altogether chose vanilla or chocolate.
(4) Twice as many children chose vanilla as chose strawberry.
(5) Just as many children chose some other flavor (not listed) as chose vanilla.

and have the class determine which sentence is being acted out. Similarly, you might give children a statement such as $4 + 2 = 6$ and have them create a story or draw a sequence of pictures that can be described by that sentence.

A particularly valuable approach for mathematical language development is making and discussing graphs—especially if the graphs describe something about the children in the class. Figure 12 shows how such a graph can promote the meaningful use of specific mathematical terms.

Solving Real Problems

In the course of a typical primary-school day, many opportunities may arise very naturally for using mathematics to solve problems that are "real" to children. When children see a need to solve a problem, they are often willing to expend a great deal of time and effort to solve it—even if it is more challenging than any problem in their mathematics books. The teacher plays a central role not only in making it possible for the children to spend time on real problems that arise, but also in serving as a facilitator to encourage such useful problem-solving techniques as understanding the problem, relating the problem to a simpler problem, encouraging different ways to solve the problem, and checking the solution.

As an example, one morning a second-grade class received a large roll of corrugated cardboard. The teacher was told to share the cardboard with the other three teachers at her grade level. She asked the children to help her decide how to do this. First, she tried to clarify the problem. How many teachers had to share the cardboard? She listed the names of the teachers and the children agreed there were four teachers. How should it be shared? One child suggested each teacher could cut off a piece, but another child realized that might not be fair. One teacher might be greedy and take too much. How could the cardboard be shared fairly?

The teacher suggested solving a simpler problem. She gave the children strips of paper and asked *them* how they might cut the paper so that there were four pieces all the same size. (They had used models like this for fraction work.) The children came up with two ways to do it (fig. 13). They then tested the two ways.

The teacher then asked the children how they might go about cutting the cardboard into four pieces so that each teacher would have the same amount. The children suggested folding the cardboard as they did the paper. Two children were selected to fold the cardboard while a third did the cutting. They soon realized that method 13b (folding in the same direction twice) was the easiest method to use with this roll of cardboard. Once they had the four sections, the teacher asked how they could show that the sections were the same size. The children placed one section on top of the other, showing that all four sections were exactly the same size. A few of the children then delivered the sections of cardboard to the other three teachers, assuring them that the roll of cardboard had been shared fairly.

This problem was probably more meaningful to the children than any textbook problem and, because of the skill of the teacher, the children were guided into the use of effective strategies.

Recognizing Alternative Solutions

Another positive approach a teacher can use to promote problem-solving ability is to make a greater effort to

recognize and reward alternative solutions to problems. In a typical classroom, it often happens that a child has devised a "different" way for carrying out a mathematical operation or solving a problem—different, that is, from the way the teacher expects the class to solve the problem. By taking the time to consider these different solutions, to give encouragement, and to praise such attempts, the teacher is sending the message that *how* you figure out a problem is important and that you can be a hero by coming up with your own way of solving a problem. It is a valuable experience for children to realize that they can have their own ideas about how to solve a problem and that they don't always have to agree with the teacher.

In one primary class, the teacher asked the children to cover faces of boxes with oaktag squares and count how many squares it would take altogether to cover the top and the bottom of the box. One little girl finished quickly and the teacher encouraged her to make a picture of what she did using square gummed stickers. A second child was asked by the teacher to do the same thing. When the two children finished, they came up with very different pictures (fig. 14). The teacher recognized the first solution of 24 as correct but was completely puzzled by the second child's response. Why had that child used the pieces of gummed stickers on the bottom? Upon questioning the child, the teacher found that the child had indeed solved the problem—in a different way. The first child covered the top of the box, found that it took 12 squares altogether, and included another 12 squares for the bottom of the box. The second child found it took 12 squares to cover the top but fewer than 12 squares to cover the bottom. He explained to the teacher that the top of the box *had to be* larger than the bottom or the top wouldn't fit on the bottom.

Without the willingness of the teacher to investigate the reason for the unexpected solution, this very sound solution would have gone unnoticed (and uncommended). It takes more time to consider different solutions and to attempt to understand how children are thinking, but it is time well spent if the children begin to realize that the teacher cares about *how* they got an answer and not just whether they got an answer.

Fig. 13

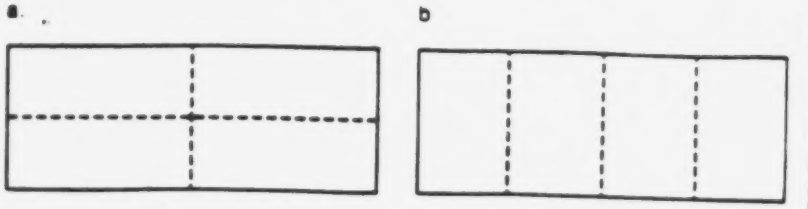
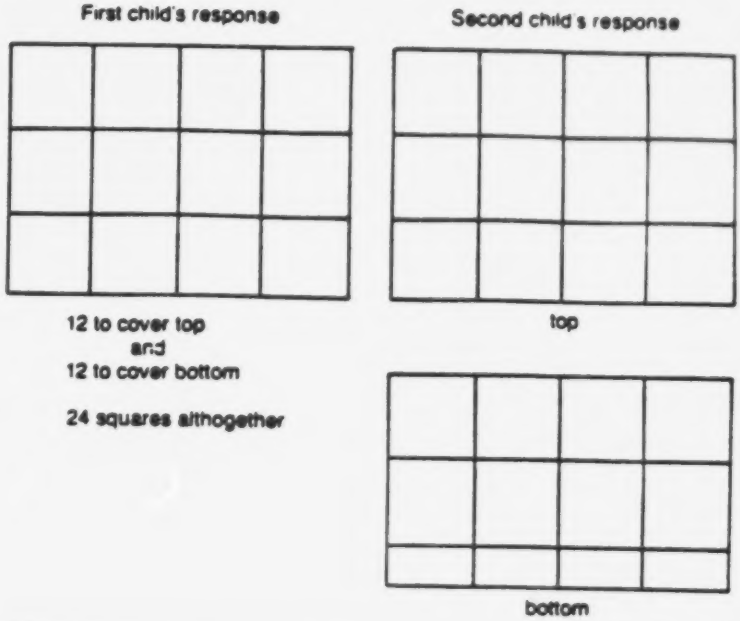


Fig. 14



Developing the problem-solving ability of primary-school children is not just a matter of having the right set of materials for classroom use. It is highly dependent upon the ability and willingness of the teacher to transform typical classroom activities into opportunities to enhance problem-solving skills. By asking more open-ended questions, looking for patterns, using spatial thinking, developing mathematical language, solving real problems, and recognizing alternate solutions, you can help children develop problem-solving skills as an inherent component of your mathematics instruction.

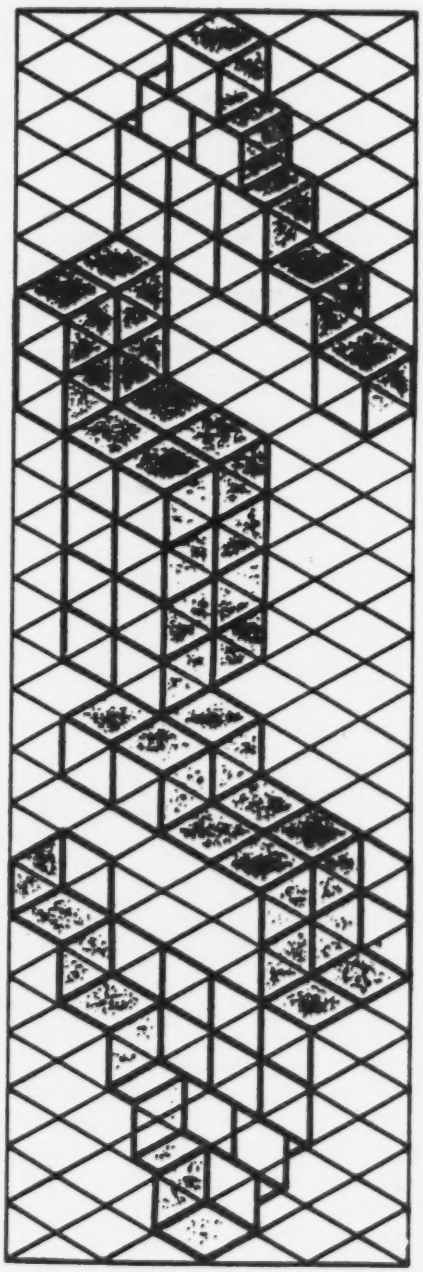
Problem Solving in the Intermediate Grades: Helping Your Students Learn to Solve Problems

By **Joan Worth**

This article is written for the intermediate grade classroom teacher because your students are at a crucial stage in the development of mathematical skills and attitudes. Intermediate graders are both ready and able to learn to solve problems and to become proficient problem solvers.

The first, most important, and most difficult (because it's attitudinal) idea to get across to students is what a problem is. Students must accept that a problem is a situation for which they do not know how to get an immediate answer. Too many students believe it is imperative that they find *the* single, correct answer immediately, and if they *cannot find it immediately* they never will. This mind-set must be changed so the pupil whose first response to a problem is a raised hand and cry of, "I don't know how to do this!" will accept your reply of, "Of course not! If you knew how to do it, it wouldn't be a problem—and we're practicing solving problems."

Once students are convinced of this idea, you as a teacher have a "double-edged" commitment to them: you must provide explicit, carefully planned and structured problem-solving instruction, and you must provide the necessary time for solving a variety of problems during regularly scheduled class sessions. To provide the time, you will probably have to reprogram what you do during your mathematics class. To provide the instruction, you will likely read and generally search for ideas to help you plan and structure your instruction. This article is designed to suggest a practical approach you might consider to help your students learn to become better problem solvers.



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General Framework

An overall procedure for attacking a problem can provide a starting point and a sense of security for your students. Indeed, it is known that successful problem solvers do follow a general plan or use many of the same steps as they solve a variety of problems. Several lists of stages for solving problems have been proposed (Suydam 1980), many based on Polya's (1973) *How to Solve It*. The specific list used is not as important as the fact that your students be equipped with some general framework they can apply in problem-solving situations. Such a framework will not only give your students a place to begin, but if they are having difficulty, it will also give you more specific information about where the difficulty is. (Caution: This is not an attempt to give students a fixed sequence of stages that they must always follow rigidly. Students will not always use all the stages or use them in the order given; the stages used will depend upon the problem and the individual.)

One general framework, used in the Iowa Problem-Solving Project and discussed by LeBlanc (1977), has four

steps:

1. Get to know the problem.
2. Choose what to do.
3. Do it.
4. Look back.

Specific Strategies

Once they have a framework, your students need specific strategies to apply within this framework. You can make several different suggestions for each stage after which you and your students can choose the most appropriate suggestions for a given problem. Initially, you ask the questions or make the suggestions. Gradually, as your students become more comfortable with the process, they should do the asking and suggesting. The ultimate goal is that your students themselves learn to ask appropriate questions.

Get to know the problem

The major purpose of this stage is to make your students think about the problem before picking up a pencil and computing, a purpose not often achieved with the traditional "read carefully" or "think hard" instructions. Ask questions or make suggestions that will focus attention on the information given in the problem, on any conditions, and on the question being asked:

Are there any words you don't know?

For a specific problem, the questions might be,

What does _____ mean? or
What is a _____ ?

Tell the problem in your own words.

This involves identifying what is known, and what question is asked.

What is the important information in the problem?

If appropriate, students could underline. If from a textbook, make a list.

What is the question in the problem?

Can you make a diagram of the problem?

Can you act out the problem?

How will your answer be labeled?

Choose what to do

This is the planning stage during which your students should consider strategies for solving the problem. Different students may choose different strategies. Initially, you will likely have to suggest something other than computation, but after you have worked together on problems using various strategies, your students should be able to suggest and choose their own strategies. The key here is that your students will not choose any that they do not know how to do, or that they do not accept as legitimate problem-solving strategies. You will have to plan specific instruction to develop skills such as writing equations; organizing data by making lists, charts, tables, or graphs; simplifying problems; and estimating. As new skills are learned, they can be added to a list of strategies. The following are some strategies your students should have to choose from, to use either singly or in combination:

direct computation
write an equation
draw a diagram
guess and test
make a simpler problem
make a list
make a table
make a graph
do an experiment

Do it

This stage is obviously very closely related to "choose what to do," but it is listed separately to highlight the importance of the planning step. Too often in the past the whole focus of problem solving was on this "do it" stage, with the emphasis on computing to get the right answer. In this framework, the "do it" stage is merely the carrying out of the plan. To "do it," however, your students need basic skills to use with the strategies listed for stage two. If any of your students are having difficulty in some aspects of computing, they should be able to use calculators so the lack of computational skill does not get in the way of the ability to solve problems.

Look back

This is the stage often described as most important and effective in helping your students become better problem solvers. Stopping with the "do it" stage, when an answer is obtained, gives undue emphasis to that stage and to "the" answer. This final stage allows you to comment on important aspects of the solution process and to reinforce techniques used.

You can guide your students first to relate their answers to the question in the problem as one way of verifying that they have indeed solved the problem. Together you and they can reflect on the plans made and evaluate the strategies used so your students understand why what they did was appropriate for that particular problem. You can then extend the problem-solving experience by changing one or more of the conditions and, finally, by looking at other similar problems.

This "look back" stage is the one that can be most difficult for you as a teacher because you cannot completely plan in advance for what will happen. The reflection, the focus, and the extension will all depend on the dialogue beween you and your students. The following are some general types of questions and statements you could use:

What were you thinking when you planned your solution?

Describe your solution to the rest of us.

Does your answer fit the question in the problem?

What other strategies could we have used to solve the problem?

Does this remind you of any other problems?

You might also change one of the conditions in the problem and ask,

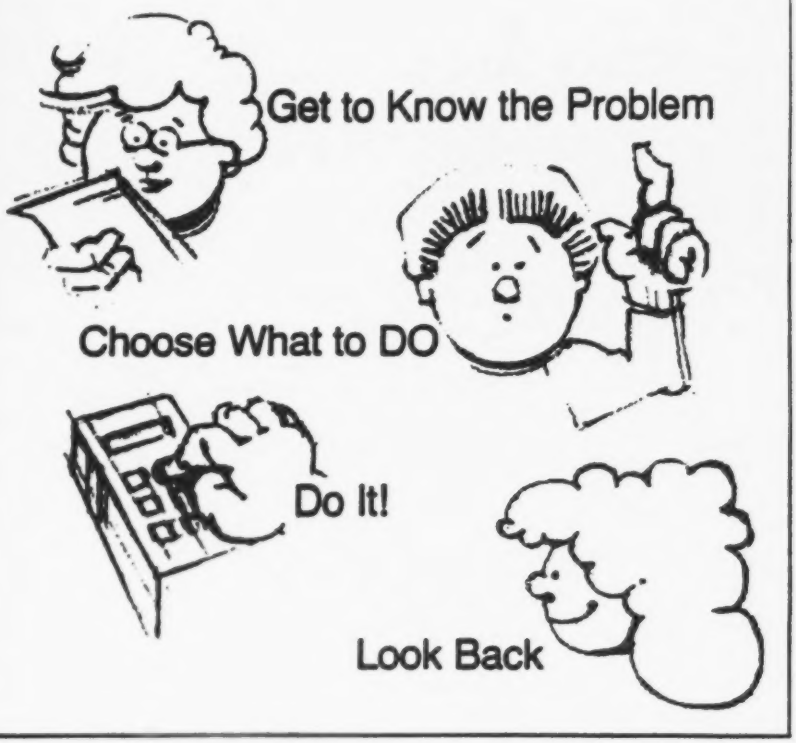
Suppose _____ , then what might we do? or

What if _____ ? Could we use the same strategy?

Presenting and Using the Framework

One way of presenting the problem-solving framework to students is to

Fig. 1



put it on a large chart, which could be posted and used as a reference (fig. 1). Then solve several problems with your students by working through each stage of the framework. This procedure not only provides an introduction to the framework, but also gives you the opportunity to act as a model problem solver for your students. As you work through each stage, ask the questions or make the suggestions outlined earlier in this article. For each stage you might begin a list of these questions and suggestions, adding new ones as you present them or your students suggest them. If these lists and the framework chart are always available for your students to use as a kind of "checklist" for themselves, it will help reinforce your use of the questions. Remember that this and the necessary practice are aimed at the ultimate goal of making your students independent, not only of the charts but of you.

When you are working through the framework, you will likely focus on one or two questions or suggestions at a time. For example, to introduce the idea of making a diagram at the "getting to know the problem" stage, you might use a version of the familiar log-cutting problem:

> Bryan can cut a log into 3 pieces in 6 minutes. At this rate, how long will it take him to cut a log into 12 pieces?

A diagram will focus attention on the importance of the number of cuts rather than the number of pieces. You could work through two or three other problems for which drawing a diagram is a good way to get to know the problem, and then add this strategy to your list for the first stage. It is important to then do at least one problem that uses a previously learned strategy. For stages two and three, two problems you could use to focus on making a simpler problem and doing an experiment appeared in the April 1977 "IDEAS" section of the *Arithmetic Teacher* (Ockenga and Duca):

> How long does it take to pass a squeeze from New York [illegible] Angeles (or Vancouver to Halifax)
> How long would it take you to tear a piece of paper for everyone in the United States (or Canada)?

The experiment with a simpler problem would involve your students in passing a squeeze or tearing paper, first just for your class, then your school, town, city, state or province. Again, add these strategies to your lists for stages two and three and follow up by solving one or two problems using previously learned strategies.

Following the presentation of the framework and some beginning, specific strategies for use within the framework, you must provide regularly scheduled problem-solving sessions—both for practice and for development of new strategies. Try for one problem-solving session a week. (This is when your commitment involving the reprogramming of classroom time, mentioned earlier, must be met.) This not only allows your students the time necessary to develop some proficiency in problem solving, but also indicates to them how important you believe it is. During many of your practice sessions, allow your students to work in pairs. This will help them to learn to express their own positions more clearly and to consider the opinions of others. Also, allow them to use calculators when appropriate.

For presentation of more difficult problems, which take a longer time to solve, let your students help you prepare a problem-solving bulletin board. A very simple format needs three sections: one to display the problem; one to display hints; one to collect solutions. In the hints area, you can display "get to know the problem" questions specific to the problem, and "choose what to do" suggestions. Students' solutions could be put into a big envelope, available for other students to look at and read. You could post the problem one day, then post hints, one or two at a time, over the next few days. The "look back" discussion could then take place during the regular problem-solving session. There is an excellent description of using a problem-solving bulletin board

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in the 1980 NCTM Yearbook (Jacobson, Lester, and Stengel 1980).

Problem Resources

Once you have made the commitment to provide instruction and time for problem solving, you might well ask, Where do I get the problems? Fortunately, the recent interest in problem solving has resulted in the publication of a variety of problem-solving resources, most of which contain some good problems. Some of these are listed in the references for this article and should help you to get started on a collection of problems. You should also examine your textbook and teacher's guide for nonroutine problems, often called "Time Out," "Think," or "Extras for Experts." Some of the more typical textbook problems may also be modified and used. The *Arithmetic Teacher* for the past few years has published several articles and "IDEAS" sections that include examples of problems. The 1980 NCTM Yearbook *Problem Solving in School Mathematics* (Krulik and Reys 1980) is a rich source of a wide variety of problems.

Your own students can also be one of the best sources of interesting problems. Making up their own problems can provide students with considerable insight into the problem-solving process. Their early efforts may often contain too much or not enough information and sometimes unrealistic numerical data, but all of this can provide for some excellent opportunities for discussion. You might start your students off with pictures from magazines or old texts and with articles and photos from newspapers. The resulting problems can be put on file cards or on stencils with the "author's" name prominently displayed. The interest your students show in writing good problems and in solving classmate's problems will be as exciting for you as for them.

There is nothing inherent in the framework or the strategies suggested here to guarantee that using them will make your students better problem solvers. It is the interested, energetic, and enthusiastic teacher who makes the commitment to provide instruction and time that will ensure rich problem-solving experiences for students. You are the only one who can take problem solving out of the arena of talks and articles and into the classroom. Thus, this article has been written to you, the intermediate grade teacher, because as Jeremy Kilpatrick (1980) reminds us, "any curriculum change begins and ends with the teacher."

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Let's Do It

Promoting Mathematical Thinking

By **John Van de Walle,** *Virginia Commonwealth University, Richmond, VA 23284*
and **Charles S. Thompson,** *University of Louisville, Louisville, KY 40208*

Let's Do It—activity approaches to teaching and learning mathematics in the primary grades, with extensions through grade six.

As we begin the fifth year since *An Agenda for Action* was published by NCTM (1980), few will argue that children's ability to think through a new and unusual situation—to reason through a problem and express their own conclusions—is a skill that is beginning to take on paramount importance in the curriculum. Today's students are clearly in an era where thinking is a more important goal of mathematics than the computational skills that we require them to practice again and again.

Many activities can be done with children, beginning in kindergarten and continuing through the grades, that are stimulating and fun and that require them to think. We believe that having children complete these types of activities contributes to the development of their general problem-solving skills.

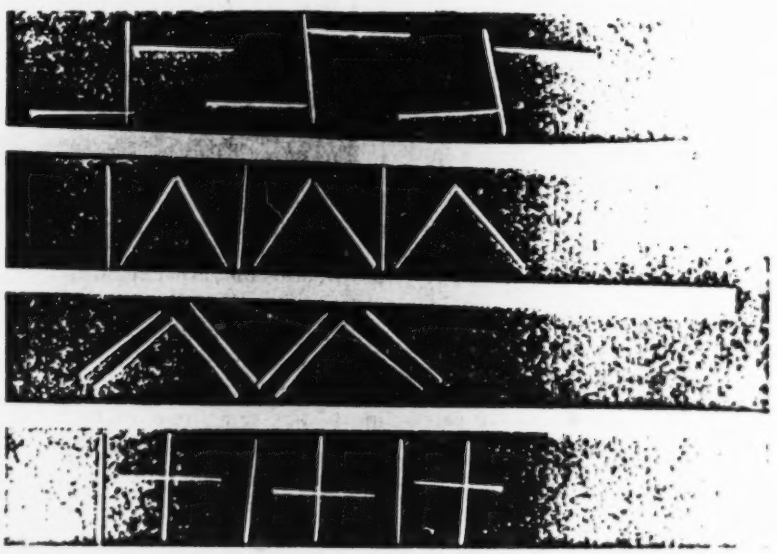
Thinking through Patterns

Patterns can be explored, extended, and created using a wide variety of manipulative materials.

Toothpicks

On an overhead projector, place some flat toothpicks in a row to make a pattern. A few ideas are suggested in figure 1, however, the possibilities are limitless. After you have placed enough toothpicks to establish a pattern, allow children to suggest how to place each additional toothpick in the pattern. Try several different patterns in this manner. To give additional assistance to children having difficulty, try "reading" a pattern to give it a verbal form: "Up, flat, slant, slant, up, flat, slant, slant, ..."

Fig. 1 How can you continue the pattern?



Next, provide children with toothpicks and sheets of dark-colored construction paper. Encourage the children to create their own patterns and extend them across the paper. Share ideas by displaying some patterns on the overhead projector. Eventually have children glue their toothpick patterns on the paper to make them permanent. The children should hold each toothpick in the center and dip only the ends into white glue. This technique is neat and tidy, and the toothpicks will still be firmly glued to the paper.

Paper strips

A significant advantage of having children make their own patterns is the internal reinforcement they receive from extending their patterns as far as they wish, repeating an idea over and over. Simple color patterns made from rectangles of colored construction paper permit these extensions to be quite dramatic. Give students an ample supply of colored rectangles

(about 4 cm × 8 cm) and let them create such patterns as red, yellow, yellow, green, red, yellow, yellow, green, ... To enable the children to extend their patterns, build the patterns on a long table or on the floor. If the pattern pieces are overlapped slightly, they can easily be stapled together and extended for several meters or even farther. Colorful patterns can then be displayed around your room—across the tops of chalkboards and around doorways and windows. In spare moments, the patterns can be "read" aloud by the children as a group. This activity will reinforce the patterning concept.

Wooden cubes

Stacks of wooden cubes can be used to make several kinds of patterns. Younger children can use short and tall stacks to create a pattern such as short, short, tall, short, short, tall, ... Primary-grade children can focus on the number of cubes in a stack to create a numeric pattern: 4, 1, 3, 1, 4, 1, 3, 1, ... These same stacks can be thought of as little "buildings" and lined up to form a patterned "street." Older children can be encouraged to develop progressive patterns with stacks or designs of cubes, as illustrated in figure 2.

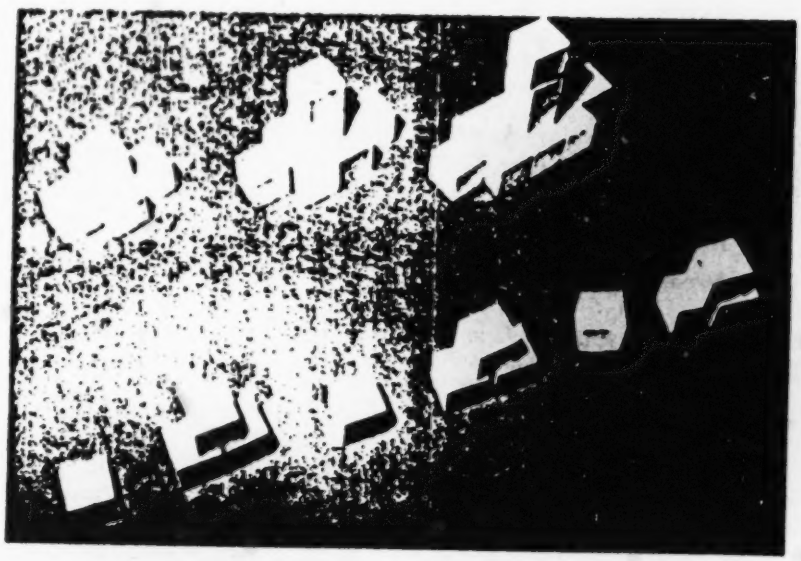
The ideas for cube patterns can be developed with a group of children or with the whole class in the same manner used to make the toothpick patterns. However, since your classroom may only have a limited number of cubes, you may want to set up a table or an area in the room for small groups of children to make and explore cube patterns. Most patterns can be recorded on paper by gluing down small squares of construction paper. For an even more realistic recording, have children glue sugar cubes together and then glue the resulting configurations on sheets of construction paper.

People patterns

Children enjoy making patterns with their own bodies. Have the class line up outside or around the room. Help the children make the line into a pattern, for example, arms up, arms down, arms out, arms up, arms down, arms out, ... Besides patterns made by positioning arms, suggest patterns made by arranging boys and girls, standing or sitting, or positioning bodies (e.g., one student bends over, another stands straight up, another squats down, one faces left or right, and one faces forward or backward).

With the children in a long line, oral or action patterns can also be developed: have each child say a word or perform some action, such as clapping or hopping in place.

Fig. 2 What comes next in each row?



More patterns

Patterning is catching. You and the students are limited only by your inventiveness. Patterns can be made with many items in the classroom or with items that can easily be collected. Here are some materials that can be used: Cuisenaire rods, shapes cut from colored paper, letters of the alphabet drawn on small squares of paper, bottle caps, buttons, Styrofoam packing pieces, fringe with colored balls, tags from plastic bread wrappers, leaves, links of plastic chain, Unifix cubes, or even paper clips.

Thinking with Attribute Items

Another problem-solving skill is classification. The process of looking for objects in a set and discovering how they are alike and different is both challenging and productive.

The activities in this section can be done with any standard set of attribute "blocks," that is, objects with the obvious attributes of size, shape, color, and, perhaps, thickness. However, three other sets of attribute materials can be used. The wide variety of properties, or attributes, of these three sets of materials makes them particularly appealing. Furthermore, these materials can easily be made or collected by your students.

Geocards

Geocards are simply squares made from tagboard or construction paper, each about 8 cm wide with a 4-by-4 grid of dots (like the pegs on a small geoboard). On each card draw a very simple shape or design using only straight lines to connect some of the dots on the card. Figure 3 shows some ideas. Keep the geocards very simple but make them differ from one anoth-

Fig. 3 Choose an attribute, then sort the cards.

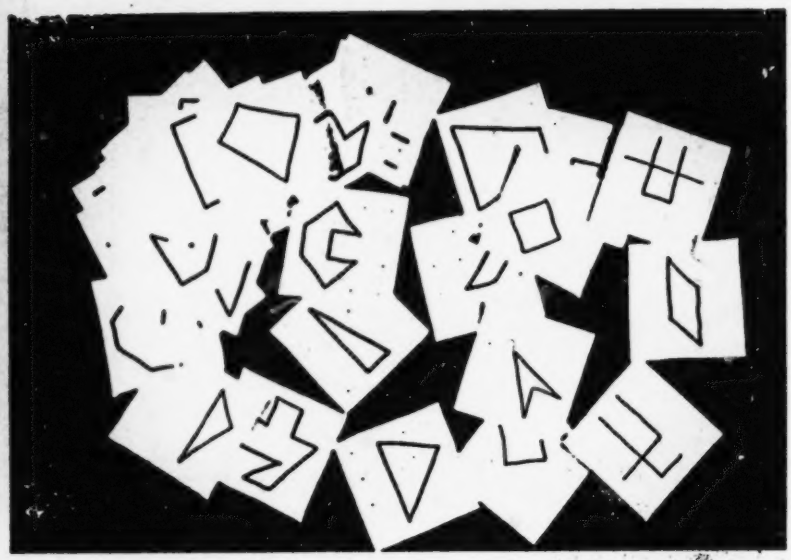
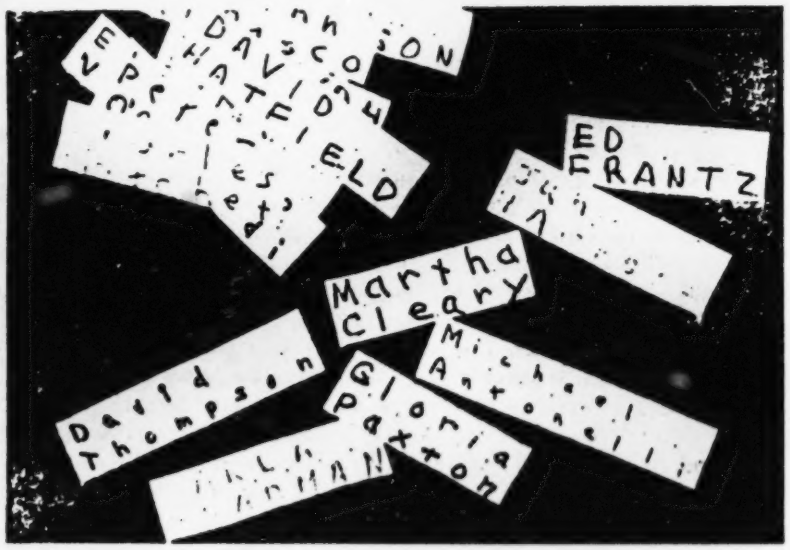


Fig. 4 Sort the names by length, number of vowels, and so on.



er. A set of fifty or more cards is appropriate. The dot grids can easily be made on a photocopier or with a template made by punching sixteen holes in a square of posterboard the same size as the geocards.

Geocards can be sorted by shape; number of straight lines or sides; symmetry (and the number of lines of symmetry); the number of dots inside, outside, or touching the figures; concavity or convexity; open or closed figures; resemblance to an object; size; or the properties of parallel sides, square corners (right angles), and so on.

Name strips

Reproduce some graph paper with 2 cm × 2 cm squares and give each child a strip that has two rows and ten columns. See figure 4. Then have children print their names on the paper, with one letter in each square and the first name above the last name, as shown in figure 4. Each child's strip can then be pasted to a piece of tagboard, or it can be laminated.

The name strips can then be sorted by length, number of vowels, number of syllables, first letter, length of first name compared to length of last name, last letter, rhyming names, girls' or boys' names, names with double letters, and so on.

Treasures

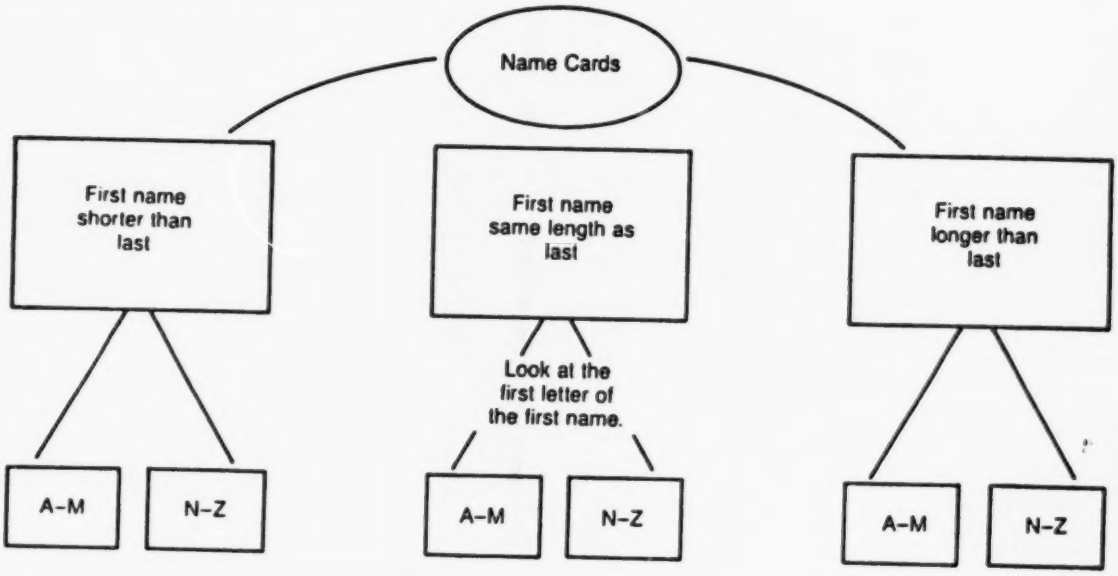
In this activity, boxes are assembled from simple materials that have a wide variety of attributes. [We call these boxes "treasure boxes." In *Mathematics Their Way*, Baratta-Lorton (1976) called them junk boxes.] The most common treasure boxes contain sets of buttons or small seashells. It is easy to create treasure boxes containing sets of any one of the following items: discarded keys, tags from plastic bread wrappers, lids from small jars, cancelled stamps, or small swatches of material. Parents may have access to unusual materials discarded by a factory or collected as part of a hobby; these items would make unique treasure boxes. The requirements are that the materials have many different attributes and that not too many materials are duplicated.

The following activities can be done with each of three materials—geocards, name strips, or treasures—and each time an activity is completed with a new material, the activity takes on a new flavor, spirit, and focus. Generally, the activities work best if small groups of children are seated at a table or on the floor. Different groups of children can work with different materials at the same time. It is not necessary to have sets of any particular materials, and since the activities can be done with any of the materials, directions can be given to the entire class at one time.

Activity 1: How many ways

You can design a variety of sorting tasks for the children. For example,

Fig. 5 One plan for sorting name cards



challenge them to find as many ways as possible to sort their materials into exactly two sets or exactly three sets. With one set of twenty keys, we have seen children sort them in more than ten ways.

Another idea is to start with at least twenty different objects and have children sort them into two or more sets. Then they can sort each of these sets into smaller subsets, using the same criteria for each set. If possible, children can find a third way to sort each resulting subset into even smaller sets. Figure 5 illustrates how this activity might be done.

Activity 2: Secret sorting

These games involve one person sorting materials without revealing the attribute or combination of attributes used as the "sorting rule." For example, the child who is the Secret Sorter begins to select items for a group, one item at a time. All the items selected have some attribute in common, but only the sorter knows what it is. The rest of the children try to guess the common attribute. If a child thinks he or she knows the attribute, he or she says "stop" and then guesses the "sorting rule." If the guess is correct, this child becomes the Secret Sorter and begins with a new secret sorting idea. If the guess is incorrect, then that child is out of that round and the sorter continues to sort the materials until the sorting method is discovered.

A similar game is modeled after the "Sesame Street" activity, "One of these things is not like the others." The Secret Sorter (or the teacher) sets out four or five objects, all but one of which have something in common. The rest of the group try to guess which object does not fit and then tell why. When guessing which one does not belong, the player must also say what the rest of the objects have in common. For example, one child might say, "Alexander Dietz doesn't belong in this group because the rest of the name strips all have first names that are shorter than the last names." Note that a guess may be a perfectly valid observation, but it may not be the rule that Secret Sorter has in mind. Points can be awarded for good observations and extra points given for finding the rule used by the sorter.

Activity 3: String sorts

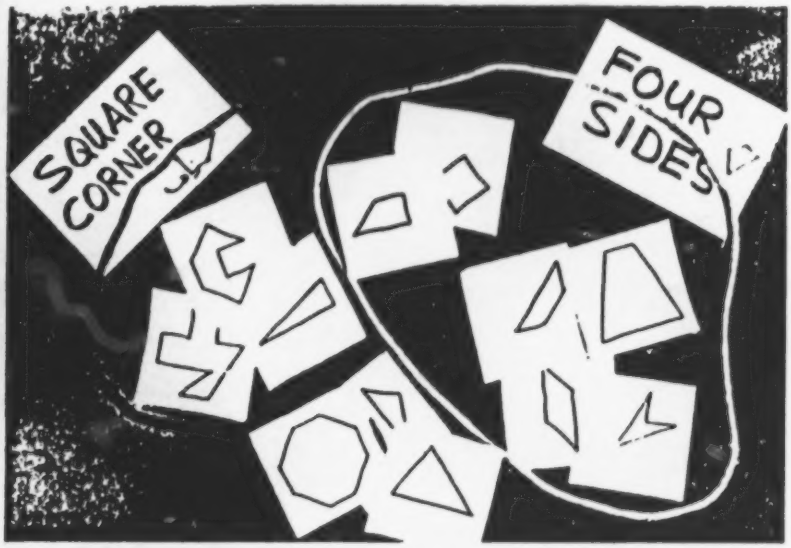
Two large loops of yarn can be overlapped to form three regions. Then objects with a common attribute are placed inside one loop. Objects placed in the other loop have a different attribute. The overlapping section contains objects with both attributes. Index cards can be used to label the attributes of each loop.

For a group activity, select two labels and place them beside the two overlapping strings. In turn, children close their eyes and select an object from the set of treasures being used. This object must then be correctly placed in one of the three sections, or outside both strings if it has neither attribute. For example, in figure 6, geocards were being sorted into "four sided" and "square corner" sets. In this case, geocards were available for all three sections and the outside. Sometimes, one or more of the sections will have no pieces in it, as would be the case with the labels "open regions" and "triangles." This sorting activity can become quite challenging if three loops are used, creating seven different regions, or if a combination of attributes is used (e.g., "four sides *and* symmetrical" or "triangle *or* square").

Thinking with Geometric Materials

Many problem-solving situations in-

Fig. 6 Using string to group the polygons



volve spatial reasoning with geometric materials. In this section we shall develop a collection of problem-solving activities that are geometric in nature, challenging, and fun.

Geoboards and dot paper

Geoboards with 5 × 5 arrays of pegs provide an excellent vehicle for children to explore geometric problems. If dot paper is available, children can also record the ideas developed when they use rubber bands on the geoboards. Kindergartners will need paper with full-sized reproductions of the geoboards. Older children can use standard 8½-by-11-inch paper with four geoboards (later sixteen) drawn on each sheet. To record designs, teach children to identify the dot corresponding to each corner of their designs. If these dots are circled first, children will find it much easier to draw lines from one dot to another.

The following are just a few challenges that you can use to get children thinking geometrically with a geoboard. Each can be modified to suit the ability and maturity levels of the children. The children's lack of familiarity with an open-ended task may require that the initial tasks ask for more specific or well-defined results. After the first few challenges we have suggested easier versions. Similar variations are possible with all the tasks.

1. How many different shapes can you make that have five sides?

a. Can you make a shape with five sides? Good! Now, copy that shape on paper and see if you can make a shape that is different but still has five sides.

2. How many triangles can you make with exactly one peg inside?

a. See if you can make a shape on your board that has only one peg inside. Look at the triangle I made. See this peg? It is the only one inside.

b. Can you make two different triangles on your board so that each has just one peg inside? Now copy those triangles on your dot paper and try to make some others that are different.

3. How many different rectangles can you make?

a. On your board make one shape that has exactly four sides. Good! Some of you have special shapes. They have square corners, just like the corner of a piece of paper. See if you can make your four-sided shape have square corners. These shapes are called rectangles.

4. Make some shapes that have two square corners. Can you find five different ones?

5. What is the smallest square you can make? What is the largest? How many different-sized squares are possible?

6. Try question 5 with triangles, right triangles, rectangles, and so on.

7. Can you make five shapes that have a line of symmetry?

8. Can you make a shape with seven sides that is symmetrical?

9. Make the biggest square you can on the geoboard. Can you cut it into two equal parts? How many ways can you do it?

10. Make some shapes with all corners square. What is the area of each square? What is the perimeter of each? Make an all-square-corner shape with a large perimeter but a small area.

11. Can you make shapes with larger and larger perimeters but all with the same area?

12. Choose two pegs on the geoboard that are in different rows and columns. How many different paths can you make from one peg to the other without going backward or away from the target peg?

13. Make a rectangle on the geoboard. Can you divide it into two regions that are exactly the same size and shape? How many different ways can you do it? Can you divide it into three congruent shapes?

The first six challenges can be adapted in some manner even for preschool children. With the teacher's guidance, all these challenges can be explored by third- or fourth-grade children. Working in groups, discussing different ideas, demonstrating new ones with examples, and gradually building the elements of a new concept or exploration are all ways that these ideas can be suited to even the youngest child.

The most difficult idea for children initially is that of the open-ended task. But with careful guidance, children will accept more open-ended activities and may even gravitate toward such tasks, since they provide the

most challenge, freedom, and room for creative thought.

If other concepts are being developed in your class, create challenges involving them. For example, none of the previous questions involve rotational symmetry, similarity, concavity or convexity, open or closed ideas, or rotations and reflections. These concepts can also be explored on a geoboard.

Cubes

Wooden cubes can be stacked in many different configurations, and this feature yields a number of possible activities. More explorations are possible with plastic cubes that snap together on any face. Consider having children do the following activities with cubes:

1. Take five cubes and build a shape. Take five more and build a different shape. How many different shapes can you build with five cubes?

2. Build a shape with eight cubes. Its volume is eight cubes. What is its surface area in square units? (Remember, to determine surface area, all a child needs to do is count the exposed faces of the shape, including those on the bottom.) What other shapes can you build with the same volume and surface area?

3. With twelve cubes, what is the shape with the largest surface area that you can build? The smallest surface area? Can you make shapes with all the possible surface areas in between?

4. Choose a number less than 30. Can you make a shape with that surface area? What is its volume? Can you make a larger or smaller shape with the same surface area?

5. Build a $2 \times 3 \times 4$ box. Notice that it can be sliced into two identical parts, one the mirror image of the other. The slice is a plane of symmetry. Can you make other shapes that have a plane of symmetry?

Simple puzzle pieces

From squares of tagboard or posterboard, some simple shapes can be cut that provide opportunities for interesting explorations and endless puzzle possibilities. Three possible patterns for cutting the squares are shown in figure 7. The idea is to have a set of pieces that are "compatible," that is, that fit together nicely or that have several pieces with edges of the same length. The exact size of the initial square is not important. If the square is about 15 cm on each edge, the resulting pieces will be large enough for kindergarten and first-grade children. After cutting one square into pieces according to the pattern, you may find it easier then to draw and cut multiple copies of the individual pieces rather than cutting them from an initial square.

Fig. 7 Some suggestions for cutting shapes from a square

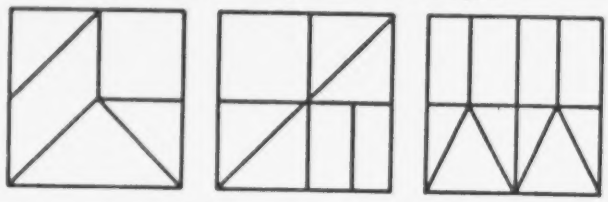
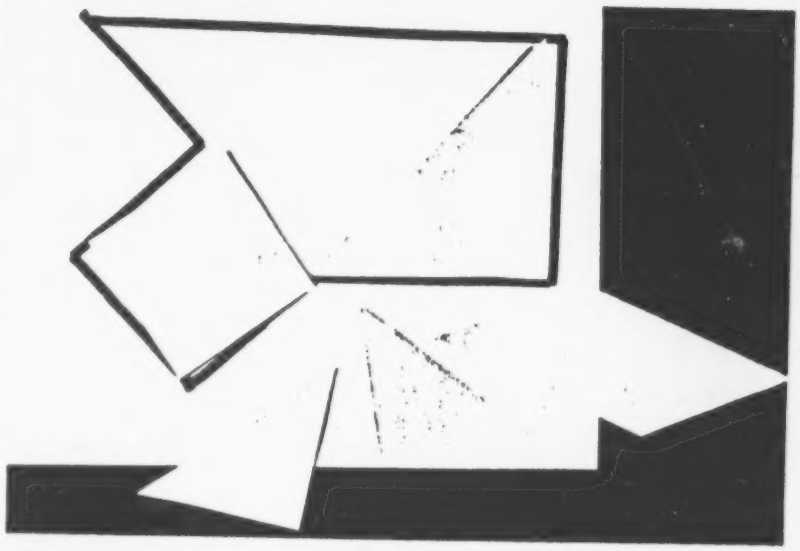


Fig. 8 How many different pieces can be used to fill in the shape?



The following activities or challenges can be completed by children using any of the three sets of pieces shown in figure 7. Another variation is to combine pieces from two or more squares or to use standard tangram pieces.

1. With the pieces from one square, how many different shapes can you make if the edges of individual pieces must match?

2. Make any shape with all the pieces from one set. Make an outline of your shape. Now try to fill in the outline but arrange the pieces differently. See figure 8.

3. Make any shape using any number of pieces. Draw the outline of the shape. See if a classmate can fill the shape. Don't tell the classmate which pieces you used.

4. Once a shape has been made, its perimeter, or the distance around it, can be measured by referring to the edges of the pieces themselves. For example, a shape can have a perimeter of two short sides plus three medium sides plus one long side. Once this idea has been established, see how many shapes the children can make that have that same perimeter.

5. The symmetry of any piece (or of any shape) can be checked with a mirror. Place the mirror on what you

think is a line of symmetry. If what you see in the mirror is exactly the same as what you see when the mirror is removed, then the line where the mirror rested is a line of symmetry. First find the pieces that have a line of symmetry. Next use those pieces to make a larger shape that has a line of symmetry. Then try to make a shape that has two lines of symmetry.

6. What familiar shapes can you make from two or more smaller shapes? Can you make a triangle? A square? A rectangle? Can you make a long skinny shape?

7. A convex shape has no "dents." Using a given set of pieces, how many convex shapes can you make?

These ideas are intended to be sketchy and open-ended. You may want to modify the directions to match the age levels of your students. If a small number of pieces are used, most of the activities can be explored even by kindergartners. By using more pieces and continually challenging students to make more shapes with a given property, you can also engage older children in geometric problem solving.

An overhead projector is an easy way to show the pieces to an entire class. If students each have a set of pieces at their desks, they can follow an idea as you model it on the overhead projector. Once a concept, such as symmetry, perimeter, or the alignment of two edges, is established, children will be able to follow up on these projects with only occasional encouragement from you.

The solutions to most of the challenges described can be copied on paper by tracing around the pieces. Doing so allows you to check the ideas that have been developed in your absence, to create bulletin boards that use the children's ideas, and to send "geometric think tasks" home so that parents can get involved in these problem-solving activities with their children.

Conclusion

In this article we have presented a few ways to involve children in mathematical tasks that require creative thinking. The intent is to reveal to children aspects of mathematics in which creative thinking is not only possible and valuable but also necessary.

Although these activities have been restricted to patterns, classification, and geometric tasks, these areas are certainly not the only ones where children can or should be encouraged to think on their own. The youngest children can and should solve simple word problems presented to them orally. Counters can be used to help them model and think through the situations. Children in the first and second grades can solve oral word problems involving multiplication and division. Children in the fourth grade and up can work on the nonstandard process-oriented problems frequently discussed in recent mathematics-education literature.

Another context for creative thinking is computer programming, either in BASIC or Logo. Both languages require and promote creative thought in a problem-solving context. Further, logic games, such as Mastermind, abound and many have been translated into computer software.

In summary, many activities are available to involve students in creative thought. We hope the ideas in this article will get you started toward promoting creative thinking in your classroom.

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Probability

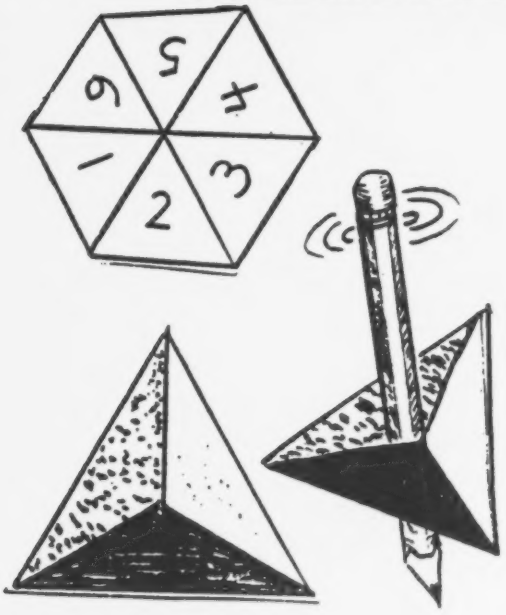
Statistics

and

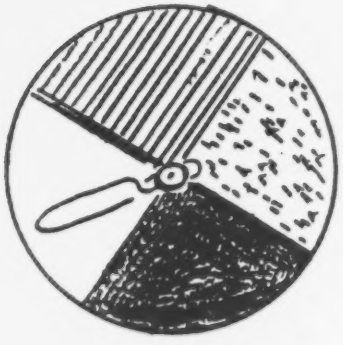
Graphs

Resources

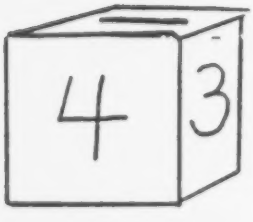
CHOICE DEVICES FOR THE STUDY OF PROBABILITY



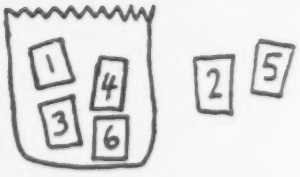
Cut a hexagon or triagle from cardboard. Push a pencil through the center. Give it a spin!



Use a paper clip and a paper fastener to make a spinner.



Write the numerals one to six on each face of a cube.



Draw numeral cards from a paper bag

GRAPHS, PROBABILITY AND STATISTICS:
A RATIONALE

Teachers of mathematics have frequently placed the greatest instructional emphasis on computational skills, seldom exposing students to the fascinating and wide-ranging areas of probability and statistics which can provide meaningful applications and enrichment within the framework of the existing curriculum. Opportunities abound for independent explorations, the handling of data, reinforcement of basic computational skills and logical thinking. Additionally, students become adept at working in groups, discussing results, comparing outcomes, creating graphs, all of these activities directed toward solving problems. Also, the use of materials such as coins, dice, spinners and cards, which are associated with fun and games, help develop a positive attitude toward mathematics.

CHANCE

Chance is involved in many games children play; spinning a spinner, drawing a card, or rolling a die are examples. If these were the only events to concern us, one would not assign the study of chance a high priority. However, life is filled with events which happen by chance or where chance appears to be a major factor. The daily choice of what to wear, meeting an acquaintance unexpectedly, a car accident; each is an event associated with chance.

As is true of algebra and geometry, some of the fundamental concepts of probability are easily comprehended at the elementary school level.

Common Misunderstandings: Nothing but chance affects the results of any one trial. Surprisingly many children, particularly younger children, do not have this understanding and do not readily accept it. Some apparently believe that one color, number, or outcome is inherently more likely to occur. Other children believe that an effort of will can make a given outcome appear. Even more deep-seated among children, and some adults, is the impression that after a string of several appearances of one outcome - say eight successive heads - the chance of a different outcome on the next trial is substantially greater. Clearing up misunderstandings such as these is an important part of the study of probability.

MATERIALS AND TERMINOLOGY

When teaching probability, we will use materials such as spinners, dice, playing cards, coins, and colored discs.

We introduce and use terminology such as likely, unlikely, more likely, less likely, equally likely, certain, uncertain, possible, and impossible, tally, frequency, sample space, outcomes, trial, event, fair, unfair, odds. The relationship of this terminology to probability will become clear as you read further.

PROBABILITY OF SIMPLE EVENTS

Children will learn how to determine that a given event will occur, both by theory and through experimentation.

As students perform experiments they should:

1. Estimate the probability and record it.
2. Carry on the experiment carefully so all trials will be conducted under similar conditions.
3. Set up an adequate recording system.
4. Communicate results pictorially, using graphs where feasible.
5. Be prepared to discuss results obtained.

Terminology associated with probability of simple events:

- zero probability (an event that cannot happen)
- probability of 1 (an event that is certain to happen)
- any other probability is stated as a fraction (a chance event)

A short way of writing the probability of an event is P(an event). For example, P(Heads) means the probability that heads will appear when a coin is tossed.

Probability by Comparing Outcomes: In a situation where each outcome is equally likely to occur, the probability of a given event is the fraction:

$$P(\text{an event}) = \frac{\text{the number of favorable outcomes that are instances of the event we are interested in}}{\text{the total number of possible outcomes of the experiment}}$$

An outcome that is an instance of the event we are interested in is commonly called a "favorable" outcome, hence the probability of an event can also be written as the fraction:

$$P(\text{an event}) = \frac{\text{the number of favorable outcomes}}{\text{the total number of possible outcomes}}$$

FINDING THE PROBABILITY OF A SIMPLE EVENT BY TOSSING THE DICE

If a pair of dice is tossed, what are all the possible number combinations (possible outcomes) that may occur? Use page 3a to record all the possibilities,

i.e.,

		2,2								
	1,2	1,3								
1,1	2,1	3,1								
2	3	4	5	6	7	8	9	10	11	12

Consider the following questions:

1. How many possible outcomes are there?
2. Which event is more likely to occur, a 2 or a 7?
3. Which event(s), 2 through 12, would be most likely to occur?
4. Which event(s), 2 through 12, would be least likely to occur?
5. What is the probability that a 2 will be rolled? a 3? a 4? ...?
6. Which events, 2 through 12, are equally likely to occur?

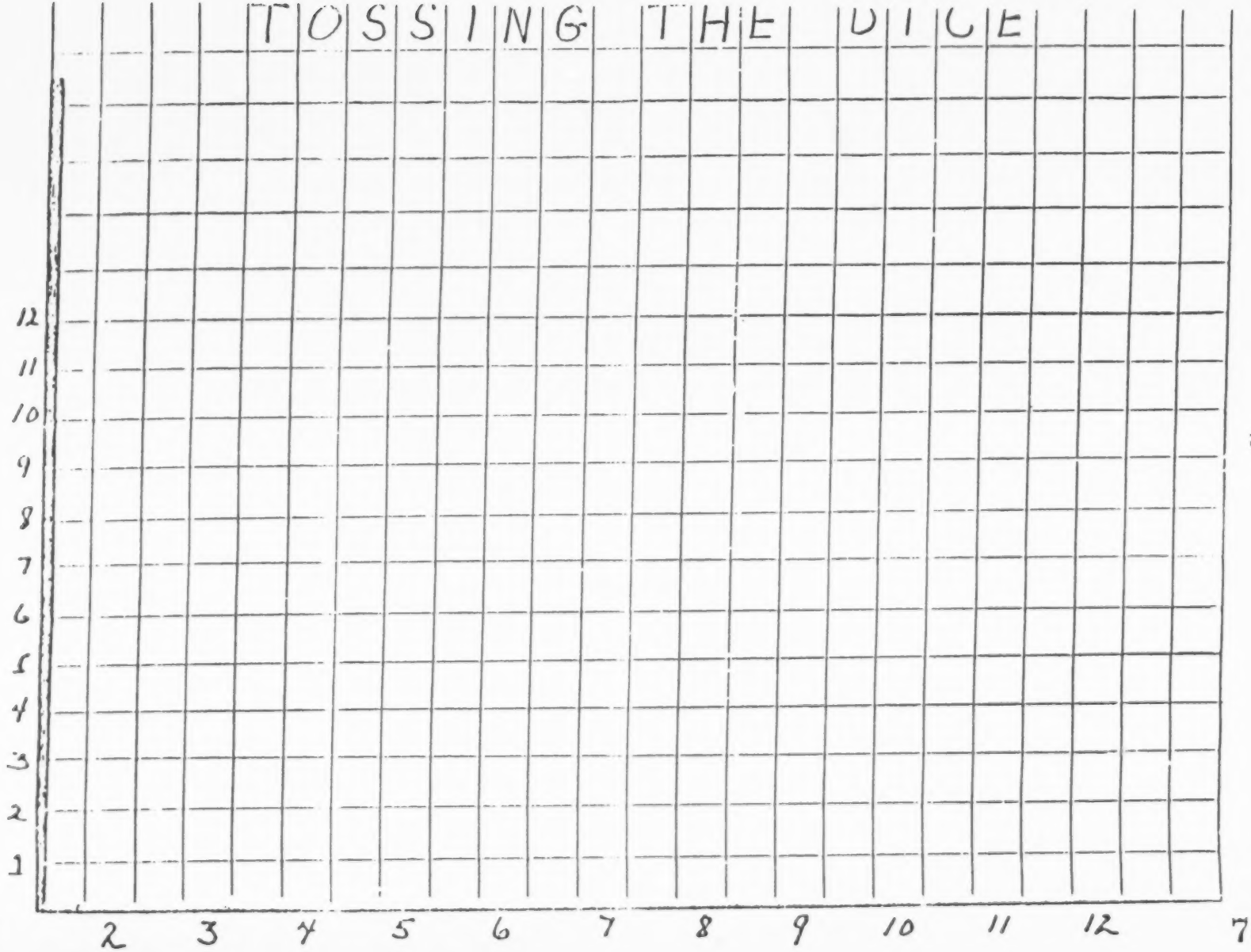
Now Experiment!

Roll a pair of dice about 50 times and use page 3b to graph your results, shading a box each time a given event occurs, i.e.,

= 4; shade a box above the 4 on page 3b.

Consider the following questions:

1. How does the graph on page 3b look as compared with the possible outcomes on page 3a?
2. What do you think would result from 100 rolls of the dice? 1000 rolls?
3. Which event occurred most frequently? least frequently?
4. Which events, if any, occurred in equal number of times? Explain
5. What computational and problem solving skills can be developed from these graphs?



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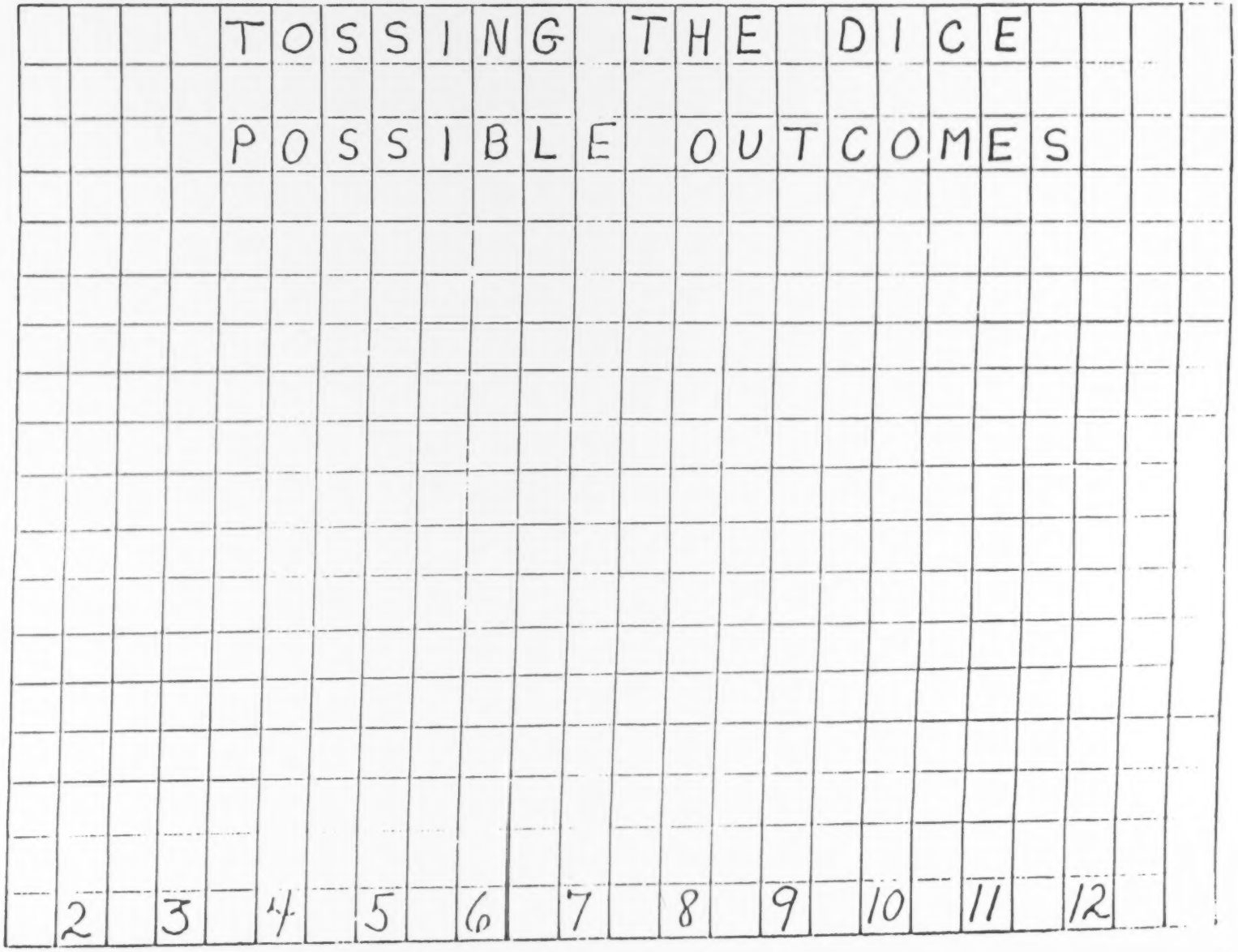

TOSSING THE DICE
POSSIBLE OUTCOMES
2
3
4
5
6
7
8
9
10
11
12

COMPOUND PROBABILITY

I. The probability of A <u>or</u> B happening is P(A) + P(B) provided the events are mutually exclusive, that is, the two events cannot happen together.

Example: What is the probability of getting a 3 <u>or</u> a 4 when a single die is rolled?

$P(3) = \frac{1}{6}$ $P(4) = \frac{1}{6}$

$P(3 \text{ } \underline{\text{or}} \text{ } 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

II. The probability of A <u>or</u> B happening when the two events are <u>not</u> mutually exclusive (that is, there is an event where they occur together) is P(A) + P(B) - P(A and B).

Example: What is the probability of getting a head or a 3 when a coin is tossed and a die is rolled simultaneously?

SAMPLE SPACE

			*			
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6
	1	2	3	4	5	6

$P(\text{Head}) = \frac{6}{12}$ $P(3) = \frac{2}{12}$ *$P(\text{Head and } 3) = \frac{1}{12}$

$\frac{6}{12} + \frac{2}{12} - \frac{1}{12} = \frac{7}{12}$

$P(\text{Head or } 3) = \frac{7}{12}$

III. The probability of A <u>and</u> B happening is P(A) x P(B) provided the events are independent.

Example: What is the probability of getting a head <u>and</u> a 3 when a coin is tossed and a die is rolled simultaneously?

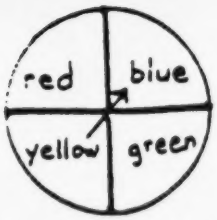
$P(\text{Head}) = \frac{1}{2}$ $P(3) = \frac{1}{6}$

$P(\text{Heads } \underline{\text{and}} \text{ } 3) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

<u>ODDS</u>

When we speak of odds we are comparing the number of favorable outcomes to the number of unfavorable outcomes or vice-versa.

Example:



The odds of the spinner landing on blue are 1 to 3.
The odds of the spinner not landing on blue are 3 to 1.

ARRANGEMENTS

Permutations refer to the different ways of arranging the members of a set, where each different order is considered a different arrangement.

Example: For the letters A, B, and C there are 6 permutations,

A B C	B A C	C A B
A C B	B C A	C B A

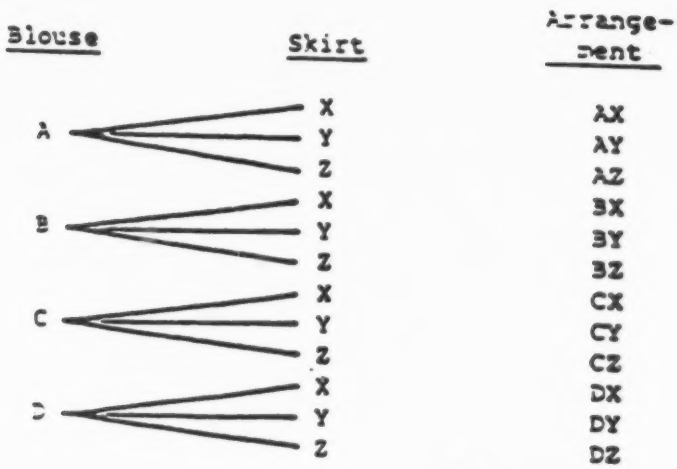
To compute the number of permutations where there are 3 elements in a group, multiply 3 x 2 x 1 = 6. (The product is called 3 factorial and is written 3!) If there are 5 objects in a group, the number of permutations is 5 x 4 x 3 x 2 x 1 = 120 or 5!

Combinations are different selections of numbers from a set without regard to the order thereof.

Example: There is just one combination for the letters A, B, and C, regardless of the order in which we place them.

Tree diagrams are another way of showing possible arrangements.

Example: Four blouses and three skirts produce twelve arrangements



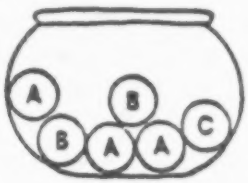
How many arrangements are possible with 5 blouses and 4 skirts? (2(

NAME _____ WORKSHEET

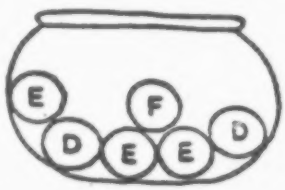
PICKING MARBLES

Suppose one marble is randomly chosen from the bowl. Fill in the letter that will make each statement true.

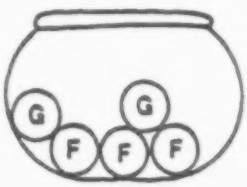
1. The odds in favor of picking _____ are 1 to 5.



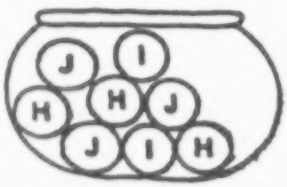
2. The odds in favor of picking _____ are 2 to 4.



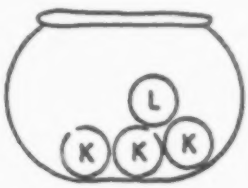
3. The odds in favor of picking _____ are 3 to 2.



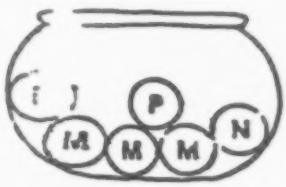
4. The odds in favor of picking _____ are 2 to 6.



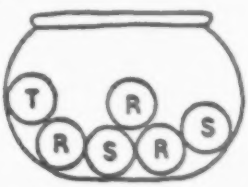
5. The odds against picking _____ are 3 to 1.



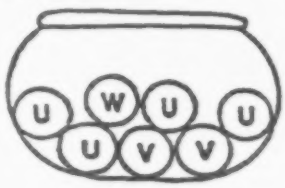
6. The odds against picking _____ are 4 to 2.



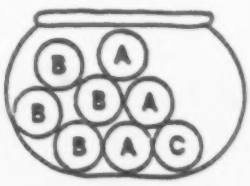
7. The odds in favor of picking R or _____ are 4 to 2.



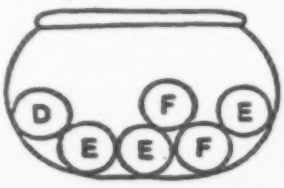
8. The odds in favor of picking _____ _____ are 6 to 1.



9. The odds in favor of picking _____ or _____ are 1 to 1.



10. The odds in favor of picking _____ _____ are 2 to 1.



STATISTICS

Statistics refers to the analysis of collected data.

The following terms are used when analyzing data.

1. **Mean** - This refers to the average of a set of numbers.
2. **Median** - This refers to the middle number in a set of ordered numbers.
3. **Mode** - This refers to the number or element which occurs most frequently in a set of data.
4. **Range** - This refers to the difference between the highest and the lowest number in a set of data.

Consider the following test scores:

65, 75, 80, 80, 80, 82, 82, 95, 99

Mean = 82

Median = 80

Mode = 80

Range = 34

Developing Concepts in Probability and Statistics—and Much More

By **James V. Bruni** *and* **Helene J. Silverman**

In recent years school mathematics programs have included an increasing emphasis on probability and statistics. Many of the activities suggested in these programs offer wonderful opportunities to involve students in active learning experiences in which they perform experiments using a variety of manipulative materials (e.g., coins, spinners, dice, and cards). These activities are highly motivational and certainly lend themselves to the use of manipulative materials. However, all too often such activities tend to become supplementary or enrichment experiences that are quite limited in scope and isolated from the rest of the school mathematics program, and the *connection* between using manipulative materials and learning basic mathematical concepts is weak.

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The activities in this article emphasize a four-step approach for using manipulative materials to develop some fundamental concepts in probability and statistics while using ideas about fractions, ratios, proportion, and percentage as an integral part of problem-solving investigations. The four steps are these:

1. Introducing the model
2. Establishing a recording system
3. Reflecting on the experience
4. Generating new experiences

The purpose of this approach is to provide a structured methodology that teachers can use to take greater advantage of the power of manipulative materials. Each experience with manipulative materials leads to a discussion to build necessary vocabulary based on the mental imagery developed (introducing the model); a transfer of the experiences to tables, charts, and graphs as a way of organizing those experiences (establishing a recording system); an examination of the records to recall the concrete experiences, identify possible pat-

Fig. 1 Keeping track of red blocks drawn

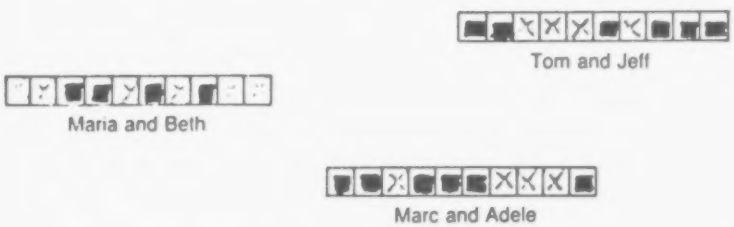
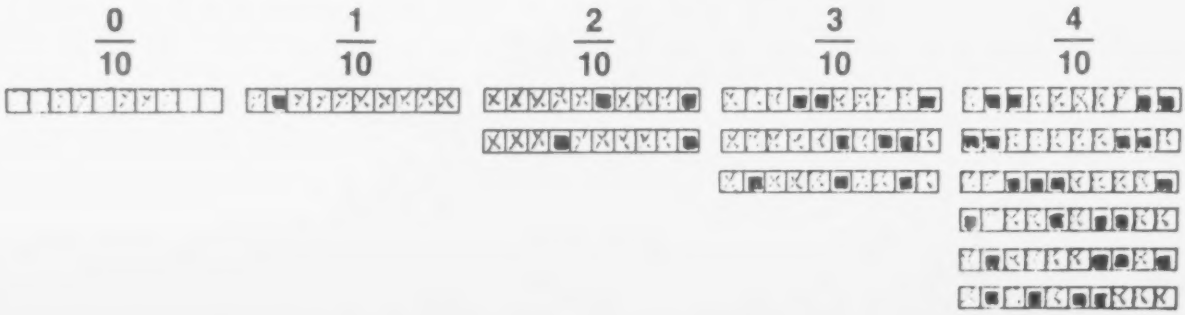


Fig. 2 Recording the number of reds in ten trials



terns, summarize information, and raise new questions (reflecting on the experience); and an exploration of related activities intended to reinforce and extend the concepts introduced (generating new experiences).

Primary Grade Activities

Introducing the model

Have students work in pairs. Give each pair a small, brown paper bag and two wooden or plastic cubes of different colors (e.g., red and blue). Have them put the two cubes into the bags. Ask if they can guess which cube they will pick if they take one cube from the bag without looking. Have them take turns guessing, shaking the contents of the bag, and taking a cube from the bag, *replacing it after each turn*. Talk about how good their guesses are. Are they usually right? How often? Which color do they get most often? How often will a specific color be drawn in ten trials? Does a pattern occur in the way a specific color comes up? Have students predict how often a specific color is likely to come up in ten trials and then experiment to check their predictions.

Establishing a recording system

Graph paper cut into strips of ten squares can be used for keeping a record. Give each pair of students a strip of ten squares. Explain that the squares will be used to keep track of how often a specific color (e.g., red) is drawn. Each square represents one turn. Have each pair of students take a cube out of the bag without looking. If it is red, color the square corresponding to that turn. If not, put an X in that square. In this way, each pair of students can keep a record of when a particular color is drawn, how often, and in what order for ten trials (fig. 1). Talk about the recorded results. Did anyone get red *every* time? Did anyone *never* get a red block? Which color was drawn more often?

Each pair of students can repeat this experiment and create records on several strips. You can collect these strips and put them on the chalkboard or bulletin board, sorting them according to how often one color was drawn in ten trials. The notation 0 of 10, or 0/10; 1 of 10, or 1/10; and so on; is useful for describing the results (fig. 2).

The power of manipulatives is enhanced by a four-stage teaching process.

Reflecting on the experience

Discuss the distribution of outcomes the group has generated. How many times did one color usually come up? With a large number of experiments, the results are likely to cluster around the outcomes 4/10, 5/10, and 6/10. Does a "pattern" exist in the order that one color comes up? Do the colors "take turns"? If one color comes up two times in a row, is the next block likely *not* to be that color? As they begin to realize that no constant pattern evolves and that the result of one trial has nothing to do with the next one, they begin to recognize the meaning of a random event and the independence of trials.

Extending the experience

This activity is meant to be just the beginning of many similar probability experiments with bags containing blocks in various color combinations. Encourage students to investigate how likely they are to get a specific color when they change the contents from two blocks of different colors to other combinations of blocks. For example, individual bags might have three or four blocks of two different colors or more blocks of one color than another in varying proportions.

As the students carry out these experiments, have them predict how often one of the colors is likely to come up in ten trials, then complete the experiment to check their predictions. In this way, the class can generate over fifty experiments in a relatively short time. Then discuss the original predictions and the actual distribution of the outcomes. Relate each experiment's outcomes to the contents of the bag. For example, if red usually came up less than half the time, were less than half the blocks in the bag red? As you develop graphs showing the outcomes of several different experiments, you can challenge the students to match these graphs to the original bags.

Intermediate Grade Activities

Introducing the model

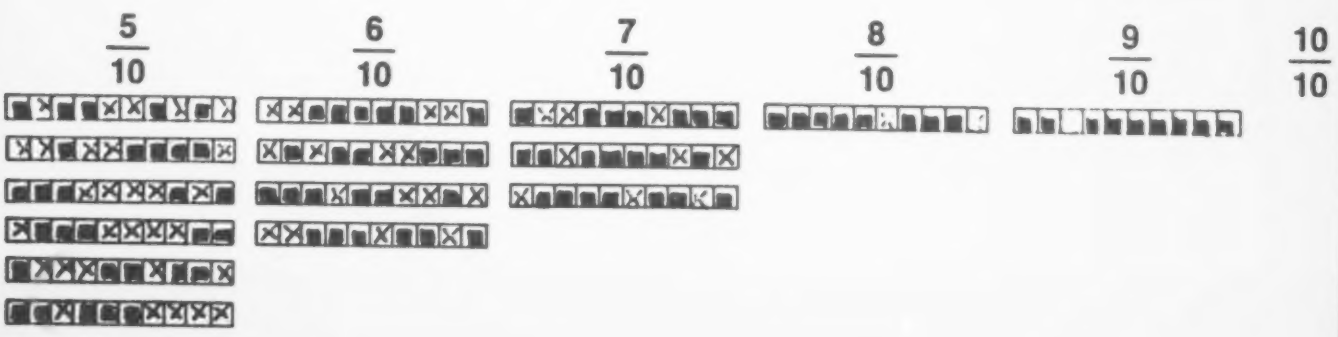


Fig. 3 Extending a table.

Andrea:

Kind	Number
lima	5
kidney	7
pinto	7
black	9
Total	28

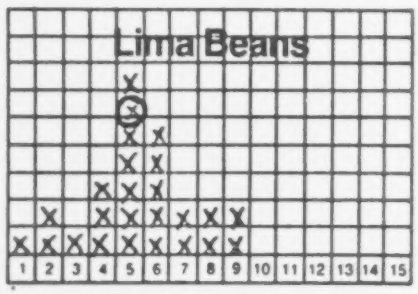
→

Andrea:

Kind	Number	Fractional part	Decimal part	Percentage
lima	5	$\frac{5}{28}$	0.18	18%
kidney	7	$\frac{7}{28}$	0.25	25%
pinto	7	$\frac{7}{28}$	0.25	25%
black	9	$\frac{9}{28}$	0.32	32%
Total	28	$\frac{28}{28} = 1$	1.00	100%

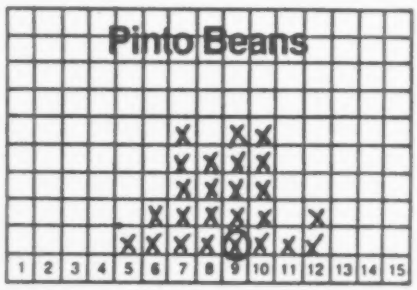
Fig. 4

Number of lima beans in each scoop: 5, 8, 3, 6, 5, 9, 4, 7, 6, 5, 8, 2, 6, 2, 9, 1, 5, 7, 6, 5, 4, 6, 4, 5, 5



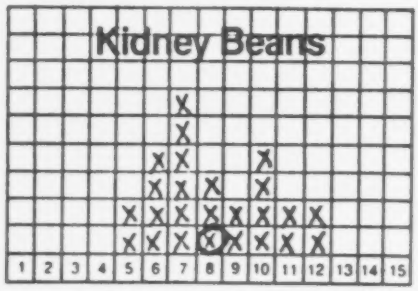
Range 8
Mode 5
Median 5
Mean 5.32

Number of pinto beans in each scoop: 7, 8, 9, 9, 7, 6, 8, 10, 10, 12, 12, 7, 9, 10, 8, 9, 7, 10, 5, 11, 9, 10, 7, 6, 8



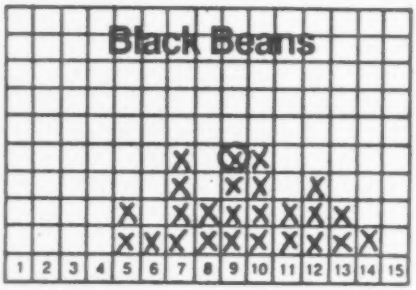
Range 7
Modes 7, 9, 10
Median 9
Mean 8.36

Number of kidney beans in each scoop: 7, 10, 6, 12, 8, 8, 9, 7, 7, 7, 11, 9, 10, 6, 5, 7, 6, 10, 10, 12, 11, 5, 6, 8, 7



Range 7
Mode 7
Median 8
Mean 8.16

Number of black beans in each scoop: 9, 7, 9, 10, 11, 12, 13, 5, 7, 10, 7, 10, 12, 13, 9, 9, 8, 10, 14, 5, 7, 6, 12, 8, 11



Range 9
Modes 7, 9, 10
Median 9
Mean 9.36

Pour the contents of one-pound bags of four types of beans (e.g., lima, kidney, pinto, and black) into a large container and mix them thoroughly. Provide a scoop or spoon so that each student can scoop some beans into a small container (a coffee scoop works well). Before the students begin, ask them to predict how many beans of each kind will be in the scoop. Many students will notice that the sizes of the beans are different, with the lima beans largest and the black beans smallest and correctly reason that each scoop or "sample" will contain more black and fewer lima beans. Others may raise questions about the "standardization" of a scoop, reasoning that although rules can be made about how level the scoop should be, variations will occur.

Have them each sort their beans by kind and display a chart of their sample. Then collect all the samples so

Table 1
Summary of the Data

Type of bean	Range	Mode	Median	Mean
Lima	8	5	5	5.32
Pinto	7	7, 9, 10	9	8.36
Kidney	7	7	8	8.16
Black	9	7, 9, 10	9	9.36

that the distribution can be further analyzed to predict the actual contents of the entire container, the "population." Help the students to focus on the distribution of the beans by asking such questions as, "Which kind of bean do you think there will be most of?" "Why?"

Establishing a recording system

Ask each student to make a table showing the number of each kind of bean and the total number of beans in a scoop (fig. 3). Show them how to extend the information in the individual tables by comparing the number of any kind of bean to the total to obtain the fractional part. Then encourage them to use their calculators to express that ratio as a decimal. Finally, extend this idea by helping them express each ratio as a percentage. Discuss the data: for example, in Andrea's sample 5 parts of the 28 are lima beans. If 100 beans were in the sample, Andrea would expect to have about 18 lima beans—0.18, or 18 percent. Since we would expect about 32 of 100 beans in the container to be black beans, we might predict that the probability of obtaining a black bean when taking 1 bean from this collection is 32 percent.

Collect the numerical distributions for each kind of bean from the students' samples and display the data on a graph (fig. 4). Show the students how to derive some informal descriptive statistics from their data. For example, the range, or difference between the highest and lowest scores; the mode, or most frequent score; the median, or middle score; and the mean, or average.

Reflecting on the experience

Have the students summarize the data (table 1). Ask such questions as these: Why are three numbers listed as the mode for both pinto and black beans? How do the median and mean of the lima beans compare? Where is the greatest difference between the median and the mean? Which kind of bean seems to show the greatest variation in the samples? With the graphs and summarizing data, ask the students to describe the mixture of beans from the profiles using the terms *range*, *mode*, *median*, and *mean*.

Generating new experiences

Guide the students to analyze the data further. Ask such questions as the following: In one scoop of beans, it is *possible* to get more lima than black beans? Is this occurrence *likely* to happen? How many of each bean would you expect to have if you had a total of 200 beans? 1000 beans? 10 000 beans? This activity offers an application of the concepts of ratio and proportion.

As an extension, encourage the students to plan mixtures of beans with varying percentages of different beans and to collect and analyze samples from those mixtures. Can they predict which sample belongs to which mixture?

Final Remarks

Basic concepts of probability and statistics develop gradually as a result of many experiences. The activities suggested here demonstrate some ways to use readily available manipulative materials to organize such experiences. The four-step approach used in these activities can help the teacher take advantage of the power of manipulative materials in developing mathematical concepts.

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Resources for Instruction

SIGNIFICANT ASPECTS OF A MATHEMATICS LESSON

1. "Do Now" (Warm-Up) or (Seatwork)
   a. Should be related to the new lesson, not just an arbitrary exercise i.e. should include pre-requisite skills, definitions, or procedures needed for the new lesson of the day.
   b. Should be written on the board at the beginning of the lesson.
   c. Should serve as a transition between previous learnings and new.
   d. Should take no longer than five minutes to review.
   e. Should be short and attainable by students within the first few minutes of the period.
   f. Can be used to provide writing experience where appropriate by asking students to write definitions, procedures, explanations, etc.
   g. Does not have to be used in a math lesson where the lesson begins immediately.

2. Motivation
   a. Should create a <u>need to know</u> ("in order to solve this problem we must learn this new technique...") or a <u>desire to know</u> ("Why does this card trick work?")
   b. May lead from a problem or apparent paradox
   c. Is short
   d. Yields the aim
   e. Is referred to in the summary

3. Aim
   a. Should be in the form of a question.
   b. Can be easily measured for achievability.
   c. Should be written on the board and left for the entire period.

4. Homework
   a. Should be cyclical (i.e., examples from a month ago, week ago, day ago)
   b. Should be reasonable in length
   c. Should take at most 10-12 minutes to review, preferably less.
      1. Student should "carry" review.
      2. Solutions and <u>questions</u> should be written out on students' work.
      3. Board responsibilities should be assigned in advance.
      4. Can employ visual devices (overhead, colored chalk, etc.) to save time and promote clarity.
   d. Should be numbered.
   e. Should be assigned at the start of the period and written on the same area of the board or be included in a homework assignment sheet.
   f. Should include RCT math or regents review questions.
   g. Should be solved by teachers in their lesson plans.
   h. Should be checked or collected.
   i Can be referred to appropriately in the new lesson.
   j. Provides pre-requisite skills for the new lesson.
   k. May be reviewed at the beginning or the end of the period.
   l. Should include a reading assignment.

5. Development
   a. Concepts should be taught along with techniques i.e., the "why" along with the "how"
   b. One new thought should be developed at a time in the sequence of logical analysis.
   c. The problems used as models should be well chosen and graded in diffculty. i.e., they yield the desired concepts and skills to be studied.
   d. The joy and beauty of the mathematics should be revealed.
   e. Concrete models, diagrams and other visual devices should be employed.

f. The board work should be detailed, neat and written from left to right on the board.
g. Students should share a mutual discovery with the teacher.

6. Questioning
   a. Skillful questioning serves to direct activity:
      - to provoke discussion which will extend knowledge and awareness
      - to stimulate thinking
      - to measure accomplishment by testing information and understanding
      - to seek evidence of reflective thinking and judgments based on proof.
   b. Skillful questioning should keep the student more alert, more critical, more active. Therefore, frame questions of graded difficulty in order to maintain interest, and appeal to the varying abilities of students. Distribute questions, calling on non-volunteers as well as volunteers. Handle the erroneous answers of the weaker students without belittling their attempts. Encourage topical, sustained answers.
   c. To stimulate critical thinking, questions should evoke sustained answers. Therefore, it is poor practice to ask too many questions or to use only factual questions.
   d. After asking a difficult question, it is wise to pause before calling on a student. Such a pause will provide time for thought and discourage the quick, glib answer. The pause also provides time for students who think more slowly to react.
   e. Once a student begins a developed answer, permit him to finish without interrupting him.
   f. Since at all times responses should be directed to the class rather than to the teacher, other class members are free to challenge the accuracy of these responses. (such automatic procedures result, of course, from constant, patient training in the early weeks of the term and from reminders given constantly throughout the term.) Where an individual--teacher--individual recitation is used, general student participation is impossible, and lack of respect for others as well as inattention (day-dreaming, talking, etc.) will result. The teacher has to keep the group in the picture even when dealing with an individual problem.

   g. <u>Criteria for judging questions asked in the classroom</u>

      1. They should be simple and direct in language.
      2. They should be specific in directing attention to the particular problem under consideration.
      3. They should follow a planned order which develops the train of thought in a logical sequence.
      4. They should place the burden of thinking upon the student (and, therefore, should be relatively few in number).
      5. They should be so framed that the answer may <u>not</u> be a simple 'yes' or 'no'.
      6. There should be included in the lesson questions which require students to draw conclusions.
      7. Questions should be adapted to the varying levels of ability in the class.
      8. They should be sufficiently difficult to arouse effort, yet not so difficult as to preclude a response.
      9. There should be a proper balance between fact and thought questions.
      10. Questions should not be repeated. (If you must have one repeated, have a student repeat it).

7. Applications and Drill Problems
   a. Should be graded from simple problems to complex
   b. Should be reviewed on the board by students in the class
   c. Should provide enrichment for talented students
   d. Should be pre-planned (and solved completely in teacher's lesson plan).

8. Summary
   a. Should be elicited from students
   b. Often yields a "recipe" type of problem-solving procedure. Should include key concepts as well.
   c. Can be generated from a "tour" of board work from left to right.
   d. Should refer to the aim.
   e. A medial summary, elicited from the class, should be provided prior to application and/or drill.
   f. A final summary elicited from the class, should be provided after the application problems.

9. Miscellaneous
   a. Teacher should circulate around the room monitoring notebooks and assisting students with individual problems.
   b. Teachers' board work should serve as a model. They should use a ruler, be neat and write out full sentences. Diagrams should be easy to read.
   c. Teachers should have worked out solutions to all homework problems and problems used during the period.
   d. All problems used should not be test oriented.
   e. Timing of a typical lesson:
      5 minutes--"Do Now"
      5-10 minutes--Homework review
      12-20 minutes--Development
      7 minutes--Drill and/or applications.
      2 minutes--Summary

CHECKLIST FOR MATHEMATICS TEACHERS

What questions is the teacher asking?

What are students doing?

* Learn the art of questioning and encourage critical thinking.
* Master the creative use of silence.
* Get physical.
* Don't cover material -- develop concepts.
* Watch beginnings and endings.
* Use and demand correct vocabulary and notation.
* Ask.
* Motivate.
* Drill -- not overkill.
* Avoid "one-to-one".

The Art of Questioning

Effective questioning is the tool with which teachers develop logical thought, stimulate the imagination, bridge the gap between previous learning and new learning and give direction to the learning process. When skillfully phrased and strategically used, questions become one of the teacher's most flexible instructional tools. On the other hand, poor questions and poor questioning technique interfere with learning by creating confusion, discouraging pupil involvement and inhibiting the sequential development of concepts.

General characteristics of effective questions:

- They are kept short.
- They are simply phrased and direct in language.
- They are logically sequenced, the response to one question leading into the next, with each question calling for only one or two points in the chain of development.
- They are designed to require pupils to think, rather than respond instantly with an association. Whenever possible, they elicit a sustained response rather than a one- or two-word answer.
- They are frequently personalized to grasp the pupils' attention: "How would you explain to a classmate who was absent yesterday our method for changing common fractions to decimal fractions?"

Twelve Types of Questions to Avoid: A Dirty Dozen

Teachers sometimes develop ineffective questioning habits that are difficult to break unless conscious examination in precisely what kinds of questions they have been asking takes place. Ineffective questions come in many varieties, and are ineffective for various reasons. We have all heard them, and have perhaps used some of them. Here is a brief compilation of some of them:

(1) The Multiple or Overlaid Question:

Example: What is the number above the fraction bar called, and what is the number below the fraction bar called? How do we reduce fractions?

Criticism: By asking for more than one response, the teacher is limiting participation to only those pupils who know all of them. Many more pupils might be able to respond if a single question were asked at a time.

Suggestion: Ask the three questions separately, waiting until one question is answered before going on to the next.

(2) The Vague or Ambiguous Question:

Example: How about rectangles?

Criticism: The question doesn't direct pupil thinking in any particular direction. A multitude of irrelevant responses can be made.

Suggestion: What kind of angles do the sides of a rectangle form?

(3) The Choral Question:

Example: Were we right to multiply by 100? (Y-e-e-e-e-s!)

Criticism: Pupils are not made to feel personally responsible for the accuracy of their response. They are encouraged to hide behind the anonymity provided by choral response. Different responses to the same question are uttered together, creating confusion for individual pupils who may not know which one is being accepted as correct.

Suggestion: Do you think we were right to multiply by 100, ... Joseph?

(4) The Leading Question:

Example: There are 6 fours in 24, aren't there?

Criticism: A pupil who is not sure will promptly agree with the teacher without thinking about his response. He has been invited merely to nod his head in acquiescence.

Suggestion: How many fours are there in 24,... Sherry?

(5) The Tugging Question:

Example: The area of a rectangle is found by multiplying the length by the, ____...? W____, w____. Come on now; it starts with "W."

Criticism: The question reduces the discussion to a guessing game, often not even based on the subject matter but on the recall of words by virtue of their sounds.

Suggestion: How is the area of a rectangle found? OR: What two measures do we multiply to get the area of a rectangle.

(6) **The Teacher-Oriented Question:**

Example: Who can tell me how many places to move the decimal point when multiplying by 1000?

Criticism: The question contributes to a feeling that the class is organized around the teacher, and that what is most important is satisfying the teacher's requirements. It does not contribute to a group feeling of a class working together to learn.

Suggestion: Who can tell us how many places to move the decimal point when multiplying by 1000?

(7) **The Whip-Lash Question:**

Example: The unit of measure that is equal to 1000 meters is called the what?

Criticism: The question is not perceived as a question by the pupils since it does not begin with an interrogative. It sounds as though the teacher is making a declarative statement. Only at the last word do pupils learn that they are being asked for a response, and thus have no time in which to think about it.

Suggestion: Include one of the words: how, what, why, when, etc. near the beginning of every question. "What is the name we give to the unit of measure that is equal to 1,000 meters?"

(8) **The Yes-No Question:**

Example: Is the radius of a circle twice the diameter?

Criticism: Every pupil has a 50% chance of answering correctly by merely guessing "yes" or "no." The question does not encourage thought, nor does it encourage pupils to use the vocabulary that they have learned.

Suggestion: By what number would you multiply the measure of the radius to find the measure of the diameter?

(9) The Echo Question:

Example: There are four quarters in a dollar. How many quarters are there in a dollar?

Criticism: The question calls for only the most rudimentary form of response: parroting. It does not stimulate thought, and a pupil can answer it correctly without really understanding the concept.

Suggestion: Wait for some time to pass before asking pupils to recall a previously developed fact.

(10) The Question Addressed to a Specific Pupil:

Example: Robert, can you move the hands on this clock to make it read one-thirty?

Criticism: By starting with the name of a specific pupil, the teacher has turned the discussion into a one-to-one exchange. Other pupils are not encouraged to think about their responses, since the question appears to have been designed for only a particular pupil.

Suggestion: Ask the question, pause to give all pupils time to consider their responses, and then call on a particular pupil. Who can move the hands on this clock to make it read one-thirty? ... Robert?

(11) The Guessing Question:

Example: Is 25% equal to one-half or one-quarter?

Criticism: Like the Yes-No question, this type of question limits thinking and creates a situation in which any pupil, regardless of his understanding of a concept, has a 50-50 chance of answering correctly.

Suggestion: What common fraction is 25% equal to? OR: Is 25% equal to one-half or one-quarter, and explain why you made your choice.

(12) The Re-phrased Question:

Example: What name do we give a half-circle? I mean, when you draw a diameter, it divides the circle into two parts. What do we call each part? When a circle is divided into two halves, what do we call each half?

Criticism: Pupils are still dealing with the first version of the question when they are presented with second and third versions. While listening to these re-phrasings of the original question, thought is diverted and clarity is inhibited.

Suggestion: Ask one good question clearly expressed and remain silent while pupils are considering their responses to it. For example:
If a diameter is drawn in a circle, it divides the circle into two equal parts. What name is given to each half-circle?
Pupil Responses

Related to the art of effective questioning is the skill with which the teacher reacts to, or accepts pupil responses. Pupils can be encouraged to formulate their responses in such a way as to make them of value to the entire class. Teachers can further encourage pupil involvement and enthusiasm in the learning process by the ways in which they use pupil responses. Some general guidelines follow:

- Encouraging pupils to comment on each other's responses involves pupils in the learning process to a greater extent than reserving the right to comment as an exclusive right for the teacher: Do you agree with Mario's explanation? How would you correct Mary's answer? What is another way of solving Renee's problem?

- Following up an incorrect response with a sequence of questions designed to help the pupil correct his own answer is much more encouraging than merely skipping over it.

- Appropriately underscoring pivotal responses is helpful to pupils by pointing up those key concepts which are at the heart of the lesson. Recording them in the form of a summary at the board, commenting on their importance verbally, asking other pupils to re-phrase them and paraphrasing them yourself are all ways of accomplishing this.

- Refraining from repeating pupils' responses encourages pupils to address their responses to the class as a whole, rather than to one individual, the teacher. When the teacher becomes someone on whom the class can rely for the repetition of all pupil responses, it is no longer necessary for pupils to listen carefully to each other.

Interaction between pupils is minimized, and the atmosphere of a teacher-centered classroom is fostered.

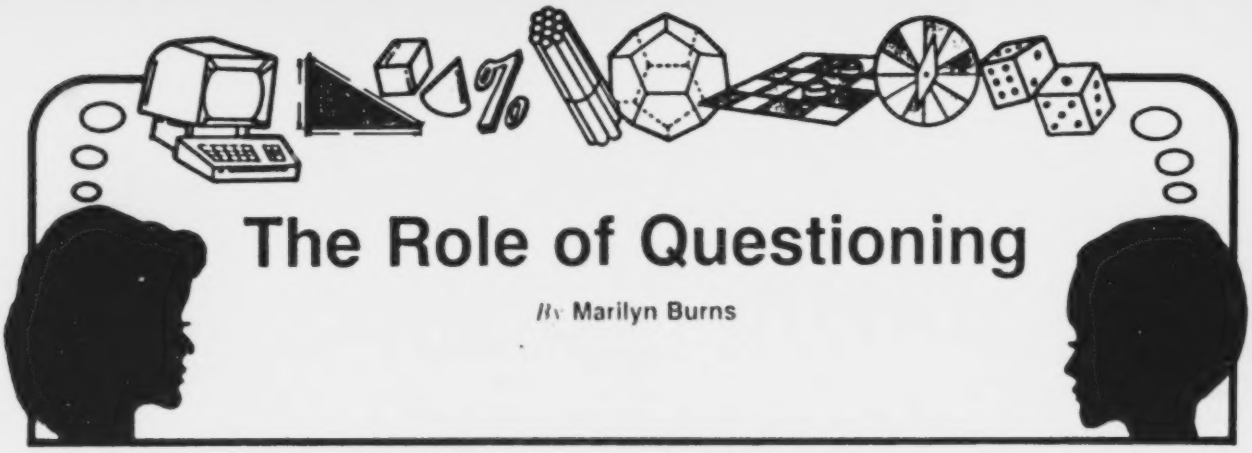
- Resisting the temptation to be immediately responsive to pupils' needs by answering their questions directly encourages wider pupil involvement in the lesson. As much as possible, re-directing individual pupils' questions for consideration by the whole class is to be preferred over a one-to-one conversation between that pupil and the teacher. There are very likely other pupils in the room who can benefit from hearing the question, thinking about their own response to it, and hearing the answer.

- Reacting to correct pupil responses in a variety of ways is more encouraging and rewarding for pupils than acting in a consistent, habitual way, such as always saying "Good."

- Remaining flexible with regard to what constitutes a correct response acknowledges pupils' contributions to the lesson more sincerely than rejecting all responses other than the particular one that the teacher had in mind originally.

- Encouraging pupils to ask as well as answer questions involves them even more deeply in the learning process. But this entails more than merely asking, "Are there any questions?" Sometimes, individual pupils have to be drawn out when it is obvious that something is bothering them but their question is not forthcoming. When pupils do ask questions, it is important that they not be made to feel embarrassed or ridiculed for having asked it. The question should be given respectful consideration by the whole class, or, at least, answered sincerely in private. The pupil asking the question should not be made the subject of a cross-examination, the surest way of guaranteeing that he will never ask another.

"Gates To Learning Mathematics," School Year Program Handbook, New York City Board of Education, Division of Curriculum and Instruction, 1981.



The Role of Questioning

By Marilyn Burns

A third grade teacher was beginning a class lesson in which the children were to work in pairs. When assembling the supplies for each pair of children, the teacher raised the question, "How many pairs of children will we have?" None of the students had a ready answer, nor did they seem to know immediately a way to proceed to get an answer. The teacher asked another question, "What do you need to know to figure this out?" One student suggested that they needed to know how many students there were. The teacher questioned further, "How can we find this out?" The children suggested different ways—having the students count off, counting by rows, or having each student in the class stand or sit down when counted.

After some discussion, when children were engaged in thinking about the problem, the teacher had them work in small groups to figure out how many pairs of students there actually were. When the children's attention was brought back to the total group, the teacher asked for their answers, continuing to prod a bit more: "How did you figure that out? Who did it another way? Maria got another answer; can you convince her that yours is right? If everyone were present today, then how many pairs would we have?"

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That the teacher seized this opportunity for a problem-solving experience is certainly a bonus for the children's mathematical learning. Involving students in real situations is always a plus for mathematics instruction. But more than that, the types of questions the teacher was asking the students to consider were helping to create a classroom atmosphere truly conducive to developing mathematical thinking abilities. Children were asked to consider questions for which answers were not immediately obvious. They were given the opportunity to connect what they did know to what they needed to examine. They were asked to reflect on their thinking. It's just this kind of skill with questioning that is needed to pave the way for teaching children to think mathematically.

Elementary school mathematics instruction has traditionally been geared toward developing arithmetic proficiency. Students spend the bulk of their mathematics instructional time learning arithmetic procedures and practicing them on worksheets or pages of arithmetic exercises in textbooks. Word problems are the usual vehicle for applying these skills, presenting situations for students to translate into arithmetic sentences and then to do the computation needed.

In traditional instruction, the primary goal is to develop computational competence. The emphasis is often on getting right answers, enough right answers to earn good grades or to do well on standardized tests. The teacher or the answer key is the source for revealing to the students the correctness of their answers. And, sadly, it's the quick right answer that is often valued more than the thinking that leads to that answer. What is missing is attention to children's deciding on the reasonableness of their solutions, justifying their procedures, verbalizing their processes, reflecting on their thinking—all those behaviors that contribute to the development of mathematical thinking.

Understanding arithmetic is important for children. However, it is essential that the development of arithmetic understanding be embedded in the context of helping children become powerful mathematical thinkers, able to solve problems, see relationships and patterns, and use numbers confidently to come to decisions. Children need to feel mathematically secure, to enjoy mathematical explorations, to be challenged and stimulated by, and interested in, mathematics. The teacher is the key to making these things happen.

The Teacher's Role

The questions teachers ask deserve careful attention. Students should be encouraged to make conjectures and to examine the validity of their thoughts. They need to search for convincing arguments that support their conjectures or that show discrepancies in others' thinking. They should be flexible enough to consider other approaches and willing to change their position if new light is thrown on the situation. Questions that promote this kind of thinking are of the following types:

What do you think?
Why do you think that?
How come _____'s answer doesn't make sense?
What if _____?
Can you find another way to explain that?
Can you convince me?

The questions will be effective only if the classroom atmosphere supports addressing these kinds of questions openly, even courageously. To create such an atmosphere, teachers need to be models of inquiring learners themselves. Students need to feel that teachers value the exploration of these types of questions as an important aspect of their mathematics learning. Teachers need to emphasize to students that the foremost goal of their mathematics learning is thinking mathematically, not merely getting right answers.

When errors in students' thinking occur, they should not be viewed as unfortunate mishaps but as opportunities for learning. The classroom needs to be a safe place so that new ideas can be tried out—so that children feel free to risk offering their thinking, even if they make mistakes. Persistence with thinking, not speed, needs to be valued. Putting the values on quick, correct responses does not help to establish an environment in which the process of mathematical thinking is encouraged.

Teachers need to ask students to find ways of verifying their solutions for themselves, rather than relying on the teacher or the answer book for verification. Verification may be difficult for teachers to demand; answer keys have long been a part of the teacher's domain in teaching mathematics. In considering this issue, however, we may be helped by taking a careful look at the real-life uses of mathematics. Real problems do not necessarily have one correct solution. At times, several possible solutions exist. Often, an exact solution isn't even required; being close is sufficient. Most important, in real life, it's up to each of us to decide when a solution is "right" or "best." In light of these issues, providing answers does not enhance students' critical thinking skills. On the contrary, giving answers can stop the thinking process. Teachers need to see their major role as focusing students' attention on meaningful questions.

Along with not giving answers, teachers need to ask students to explain their approaches and results, even when they're correct. Often, teachers nod and affirm a child's correct response but question the child only when an error has been made. If teachers question children's responses only when they are wrong, the children soon catch on. The instruction should be structured so that students describe their methods and solutions not only to the teacher but also to their classmates. Time for discussion is needed so that students are encouraged to listen, question, and learn from each other.

Providing answers does not enhance students' critical thinking skills.

A Classroom Example

The horse problem is one I've introduced to many students, from fourth graders to teachers in in-service classes. The problem always produces lively and spirited discussions. More than that, however, this problem illustrates how a teacher can work toward establishing a classroom atmosphere that distinguishes between merely getting a correct answer and focusing on the process of mathematical thinking.

> A man buys a horse for $50 and sells it for $60. He buys the horse back for $70 and then sells it again for $80. Did he earn or lose money, and how much? Or did he come out even?

When presenting this problem, I ask the students to think individually about the problem at first, without discussing it with each other. (If you haven't thought about this problem before, try doing so now.) After collecting students' answers, I have them share their thinking in small groups. Finally, I ask for volunteers to report both their solutions and the thinking that led them to their solutions.

Answers to the problem usually vary, but a core collection is always offered. Some students feel that the man earned $10, others feel that he earned $20, some argue that he lost $40, some feel he wound up $30 ahead, and some think he came out even. At times, students argue that you can't tell because you don't know how much it cost to feed or board the horse; when that argument arises, I ask the student to focus on the situation as it is rather than consider that complication. Interestingly, the same range of answers usually emerges from groups of elementary and secondary school students and teachers.

When you discuss this problem with a class, it's important to remember the goal of fostering the development of mathematical thinking. It's fairly clear to students that somehow they have to add or subtract the $50, $60, $70, and $80 amounts. It's not the arithmetic itself that poses a quandary. It's thinking about how to make sense out of the situation that is the problem. The struggle for meaning needs to be nurtured by questions that encourage the struggle:

Who has a different solution? How did you get it?

- How did you arrive at your answer?
- Did someone get the same answer but by a different way of reasoning?
- Who has a different solution?
- Can you explain a flaw in someone else's reasoning?
- Can you convince someone who disagrees with you?
- What if a man bought a horse on the first transaction and a cow on the second? Would that change the problem?

Very often, after some discussion of possible solutions, a student will ask what the real answer is. It's at this point that additional skill is required

of the teacher. Once again, it's important to remember the goal of promoting thinking. By telling the correct answer, you run the risk of having students stop their thinking. If the answer is the primary goal, students have no further reason to continue with the problem once the answer is revealed. Instead, the teacher needs to support the students' thinking. It's useful to have the students act out the problem, using slips of paper for $10 bills. It is helpful to encourage students to discuss the problem with their friends and families. Keeping mathematical exploration alive is the key.

Encouraging continued thinking may be frustrating for students. You do not want to discourage students so that they stop thinking, nor do you want to leave them with misconceptions. Making the decision not to provide an answer, but instead to keep the problem alive, is one of those judgments that teachers continually have to make. When doing so, you need to discuss your action with the students, creating an awareness of the existence and importance of mathematical thinking. This awareness needs to be nurtured consciously and continually by talking about it whenever the opportunity arises.

Final Thoughts

Teachers need to make questions of the types described a regular part of children's mathematical experience, so they come to regard these questions as a basic ingredient of their learning of mathematics. It is not enough for children merely to learn the processes of mathematics. They need to increase their understanding of what mathematics is and learn to use those processes.

It's important to remember that developing the ability to think mathematically is not a lesson objective but a long-range goal. Time is often a concern for teachers—you've got just a year to get through an enormous amount of material. Natural learning, however, doesn't happen on a schedule and often requires more time than schools are organized to provide. Discussions with questioning students take time. It's essential that teachers give the time needed for children not only to work through activities that promote thinking but also to reflect on that thinking whenever possible.

Teachers need to hold the vision of positive, open, inquiring children, who willingly explore mathematics with enthusiasm. Children's experiences in mathematics need to be exciting and fun, not in the sense of easy or trivial but in the sense of engaging them fully and keeping them involved in the investigation.

Mathematics involves a sense of investigation.

Classroom experiences must extend beyond the goal of arriving at correct answers. Children must be asked to judge the reasonableness of their thinking, to defend their solutions. Further, children's classroom experiences need to lead them to make predictions, formulate generalizations, justify their thinking, consider how ideas can be expanded or shifted, look for alternate approaches, and search for—even seek—those insights that, rather than converging toward an answer, open up new areas to investigate. Children are capable of this quality of work in school mathematics. To expect less is pointless and demeaning. Questioning is an important part of the teacher's ability to establish a classroom atmosphere conducive to the development of mathematical thinking.

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IMPROVING STUDENTS' THINKING SKILLS IN MATHEMATICS CLASSES THROUGH QUESTIONING

Skillful questions serves:

- to direct activity
- to promote discussion which will extend knowledge and awareness
- to stimulate thinking
- to measure accomplishment by testing information and understanding
- to promote student interaction

Devising Thought Provoking Questions:

When interested in developing ideas, concepts, insights, and interrelationships, teachers should use questions that ask: how, why, illustrate, show, summarize, compare, contrast, explain, describe, evaluate, etc.

Questions that promote students' thinking skills:

Explain the ...

What do you think that?

How do you think that?

How can _____'s answer be corrected?

What if ...?

Can you find another way to explain that?

Can you convince me?

What reasons or evidence ...?

How are ...similar? How do they differ?

Compare ...?

Where does _____lead you?

Why did you ...?

How did you determine that the answer is correct?

What's wrong with the statement ...?

Correcting Incorrect Statements:

By appropriate questioning the teacher can correct partially or fully erroneous statements. For example, the teacher may ask the entire class:

- With which parts of John's answer would you disagree? Why?
- What facts have we learned that tend to contradict Bill's answer?
- What's wrong with the answer just given?
- How might we revise Jill's answer to make it more accurate?

Remember, the task of the teacher is to give credit for the part of the answer that is correct and attempt to have the incorrect part of the answer improved by questioning. e.g., How could you improve the answer just given?

Questions from Students:

In order to distribute the responsibility for learning and make the class share the process, the teacher should refer students' questions to the entire class by asking: Who can help Donald by responding to his question?

QUESTIONING--COMMON MISTAKES

1. The teacher does not distribute the questions evenly among all students.
2. The teacher cannot be heard by all the students.
3. The teacher does not use English correctly when asking questions.
4. The teacher asks irrelevant or meaningless questions.
5. The teacher repeats the answer to the questions he/she asks.
6. The teacher calls on the student first and then proceeds to give the question.
7. The teacher compounds his/her questions, that is, he/she asks two questions successively.
8. The teacher asks a question which commands a Yes or No answer.
9. The teacher asks questions that are too difficult for the students in the class.
10. The teacher does not call on volunteers.
11. The teacher calls on volunteers to the exclusion of non-volunteers.
12. The teacher does not encourage the shy students to answer.
13. The teacher encourages unison answers by using questions that require only a one-word answer.
14. The teacher does not utilize thought provoking questions.
15. The teacher does not correct the incorrect English used by a student in answering a question.
16. The teacher does not grade questions in difficulty to meet the readiness of the class.
17. The teacher does not wait for an answer and give enough time for thought.
18. The teacher does not commend students for correct answers.
19. The teacher does not hold the students responsible for listening to the questions and to the answers.
20. The teacher does not encourage the students to ask questions.
21. The teacher does not use questions to emphasize or summarize a lesson.

QUESTIONING IN THE MATHEMATICS CLASSROOM

You will find below some questions which might be asked by teachers of grade 3 mathematics. Some of them are well constructed, in good form and in good English. Others are faulty in one way or another. Criticize each question according to the following pattern:

a. If you think the question is a good one, indicate this fact, and explain why you think it is good.

b. (1) If you think the question is a poor one, indicate this fact, and explain why you think it is poor, and

(2) rephrase the question in a way which will make it satisfactory.

1. Mary, what is the solution of ?+8½11?

2. How can we decide if the answer to the subtraction problem is correct?

3. Count backwards by 10, the number following 210 is what?

4. How can we compare a line segment and a ray?

5. Wouldn't you say that the perimeter of a quadrilateral is the sum of the lengths of its sides?

6. How can we decide if regrouping is necessary when adding two numbers?

7. What is the procedure for adding fractions with like denominators?

8. Is (1,4) the same as moving 1 space horizontally and 4 spaces vertically?

9. What is the quotient of $4\overline{)84}$?

10. Which fraction is larger $\frac{1}{4}$ or $\frac{1}{5}$?

11. What about negative numbers?

12. How can we round 251 to the nearest ten?

D#101

QUESTIONING IN THE MATHEMATICS CLASSROOM

You will find below some questions which might be asked by teachers of grade 6 mathematics. Some of them are well constructed, in good form and in good English. Others are faulty in one way or another. Criticize each question according to the following pattern:

a. If you think the question is a good one, indicate this fact, and explain why you think it is good.

b. (1) If you think the question is a poor one, indicate this fact, and explain why you think it is poor, and

(2) rephrase the question in a way which will make it satisfactory.

1. What is the numerator in $\frac{2}{3}$, and what is the denominator in $\frac{2}{3}$?

2. What is the procedure for rounding decimals in hundredths to the nearest tenth?

3. Is 5 a prime number?

4. How does the procedure for subtracting fractions with unlike denominators different from subtracting fractions with like denominators?

5. James, do you know how to multiply $\frac{2}{3} \times \frac{3}{4}$?

6. What about reciprocals?

7. How does a prime number differ from a composite number?

8. What is meant by a ratio?

9. The sum of the angles of a triangle is what?

10. Is -10 negative or positive?

11. Are the angles of the square each equal to 90&?

12. We compare integers, how?

D#101

ON HOMEWORK

HYPOTHESIS: Homework which reaches optimum significance needs to be carefully planned to harmonize with instructional goals. It should accent quality more than quantity. It should prevent wasted or uneconomical effort by students. It should be cyclical in nature to assure retention of previously learned material.

CHARACTERISTICS OF A POOR ASSIGNMENT:

1. The assignment consists of an arbitrary designation of pages or exercises in the text, chosen hurriedly and with little thought.

2. Prerequisite knowledge and skills have not been identified. There is no reason to believe the students can handle the material.

3. The assignment is fragmentary and bears little relation to what has gone before or what is to come after.

4. Although drill is not the goal, there is repetition to the point of frustration or boredom. No challenging exercises are present.

5. The assignment has not been planned for utilization in subsequent lessons; it doesn't set the stage for new ideas.

PLANNING VERSUS MAKING A HOMEWORK ASSIGNMENT:

Teachers should make a conscious effort in their planning:

a. to investigate future topics
b. to reinforce previous topics
c. to provide experiences that promotes discrimination among concepts in order to make the mathematics more available for the solution of the problems arising outside the mathematics classroom.

STEPS TO CONSIDER IN PREPARING CYCLICAL HOMEWORK ASSIGNMENTS:

1. The teacher should first list the topics that are to be taught each day (for the days between full-period tests).

2. The homework assignments are then planned <u>vertically</u> in the following manner: <u>ONLY TOPICS ARE LISTED FIRST</u>.
   a. the topic on the day's lesson is included for at least 3 successive days and then including it every other night or every third night, etc.
   b. The other topics chosen are from previous lessons alternating when they are given. Continuous computational topics should be included in most classes.

   c. After the topics have been listed for each assignment, the teacher should consider the length of the assignment and whether enough topics are covered for the needed variety on the test following these set of assignments. If adjustments are necessary, additional review from previous work can be included. Also movement of topics (because of too many long questions in one assignment versus not any long question, etc.) may be done at this point. Teachers are reminded that homework must be long enough to cover review but short enough to have students explain their solutions when it is gone over and yet allowing time for the teaching of a new developmental lesson each day.

d. After this planning, the books can now be used to select the problems. If this is done in a vertical fashion considering one topic and selecting problems for all of the assignments on this topic, gradation and variety on successive nights is made easier.

Each assignment should include at least 1 question that is verbal in nature.

Each assignment should contain in both single concept questions as well as multiple concept questions.

The new material should be lightly stressed in the homework. Giving 1 problem on the new topic is more successful than giving many. In this way, greater stress can be placed upon asking students to verbalize the rules, methods, or procedures needed to solve such a problem. If a student cannot do this problem, giving more than one problem will cause further frustration and math anxiety. If a student makes an error, he would reinforce this error many times if more than one problem is given. Giving one problem also provides a quick way of determining whether the new material was learned. Variations of the topic are given in successive nights.

e. Each assignment includes pages to be read on the topic taught that particular day.

f. Teachers should realize that if there is any uncertainty of completing a topic on a particular day, it is better not to include this topic on that night's assignment.

ADDITIONAL COMMENTS WITH REFERENCE TO HOMEWORK:

1. If homework assignments are planned from test to test,

   a. teachers will not only know whether they have enough topics for a varied and comprehensive test but will also know if they have both long and short topics for the given test.

   b. teachers can include review of topics previously tested that in their judgment need further review spirally in future homeworks and then include these topics on the next test. This should help the student who was unsuccessful on a given topic when it was first tested become confident when they are successful on the next test.

   c. students become more familiar with a comprehensive test and not as thrown when they take a state exam or a city-wide exam.

   d. the teacher can make up the test at the same time that they have prepared the set of homeworks between tests. It this way the test can be written, duplicated, proofread and a key prepared well ahead of it being given. Teachers are then directed as to what schedule of teaching should occur and in some way prevents going off on a tangent in class.

2. The overhead projector is very helpful in going over the homework. Students may purchase china markers at the beginning of the term. The teacher then gives an acetate when assigning problems to each student to bring in the following day. The student then comes to the machine, explains his/her work, chairing any questions asked from the class. This furthers verbalization by students and provides more sustained explanations. Teachers need to stress that students give explanations of what was one or thought of to go from step to step rather than have them read the work. No time is spent putting up the problems

on the board. All work becomes visable to the entire class. More student interaction takes place since this student is chairing questions from the class. Students may borrow the acetate and bring in the next day. If a disruption during a period (fire drill, etc.) the homework may be gone over the following day. Time may be spent on training students to pay attention to the discussion on hand not to copying of the work. More emphasis can be placed on re-doing work done incorrectly rather than mere copying of the solution. Going over the homework after the applications of the new lesson and before the summary of the lesson usually speeds up the going over of the homework rather than extending if to the entire period or most of the period.

3. Teachers are asked to prepare all solutions to each assignment in advance of their need with the students. In this way the teacher can determine the types of errors that could occur and know what to stress in the teaching of the topic. Time is saved since the teacher can quickly determine the point of error and ask questions accordingly.

4. Teachers may prepare homework sheets for distribution to the class. The benefits of having such a sheet are as follows:

   a. for students:
      (1) if absent, they have the homework and reading assignment and can come in prepared
      (2) provides information as to what topics to study for a test
      (3) gives enough advanced notice of test date to allow for planned study periods
      (4) if they know of another commitment for a particular night, they can do their homework in advance

   b. for teachers:
      (1) eliminates need for writing assignment on the board at the beginning of each class period
      (2) eliminates nightly pressure
      (3) eliminates having to first review which problems were already given
      (4) allows for gradation of problems
      (5) allows for a better planned test
      (6) provides material to be given to new admits and those students who return after extended absence.

   c. for parents:
      (1) gets better image of school, department and teacher since they see more planning
      (2) appreciates that child can do work if absent
      (3) appreciates not having last minute notice of test

   Sample model of information on homework sheet:

   #23 Given: Due:

Name of Text	104	201	204	212	277	313	Page #
Read 236-8,249, 250,305,306	46	15	48	13	19	16	Problem #

   Answers to problems:

   Note:

Putting answers on sheet helps student learn how to use answers when doing solutions and reduces math anxiety.

THE FOLLOWING MAY BE DISCUSSED WITH STUDENTS AND GIVEN TO THEM IN PRINTED FORM FOR REFERRAL AT HOME WHEN DOING THEIR MATHEMATICS HOMEWORK:

HOW TO DO YOUR MATHEMATICS HOMEWORK

PURPOSES OF HOMEWORK:

a. To help reinforce what you learned in class.
b. To help identify those parts of the lesson you do not understand.
c. To increase the number of correctly solved problems to which you can refer when reviewing for a test.

1. Read the introductory section of the assignment to identify and understand the principle it teaches.

   a. Make certain you understand the meaning of every word in the section.
   b. Visualize every step along the way.

2. Do the sample exercises.
   This will determine how well you understand the principles being taught and whether you will be able to work problems based on those principles.

   a. Read each step carefully in the model problems.
   b. Understand every term used.
   c. Follow the reasoning behind each manipulation.
   d. Solve the problem all over again on your own.

   It is a good idea to reserve a section of your loose-leaf notebook for working through the sample problems in the book.
   Writing out the steps is far better than merely reading through what the author has written.
   This section in your notebook will be a handy review for you before tests.

3. Recite the principles you have learned. If you cannot repeat the principles used, go back to the reading material prior to the exercises and read it through again.
   Write the principles and procedures used to solve each type of problem on an index card from which you can study for a test as well as quickly review previous work each day.
   Make review a part of your daily calendar. It will eliminate last minute cramming and further retention of your mathematical skills.

What to do if you "get stuck" on a problem!

a. Look back at the book, at your notes, and at the sample model problems.

What to do when an answer is wrong in your homework.

a. Analyze the problem to find out at what point the error was made.
b. Listen to the analysis done in class.
c. Take down the correct answer.
d. At home, re-do the problems you got wrong before trying the new assignment.

A Homework Guide: Winning the Game of Student Achievement

Harvey C. Foyle and Gerald D. Bailey

Every basketball team has a starting five players. These "starters" lead the team and carry out the coach's plans that are based on common sense, experience and research. The whole team consists of many additional players who are necessary and important to the success of the team. However, only five players are on the floor at the same time. Homework procedures and practices are similar to the starting players on a basketball team. Homework has numerous aspects but there are five basic guidelines that start the homework process and carry out the practitioner's plan. Based on our teaching experiences and our homework research, the following homework guidelines were developed and applied in a classroom homework experiment.

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Homework Guidelines

1. Homework should be regularly assigned. (Leonard, 1965) Students become familiar with a teacher's way of handling homework. If homework is infrequently given, students will not develop the habit of completing the homework according to the teacher's expectations. Homework does produce higher student achievement when assigned about three times per week in a subject area (Foyle, 1984). The length of homework assignments must be carefully considered. At the secondary school level, twenty minutes per homework assignment would be sufficient in high school subjects and fifteen minutes in junior high subjects. At this rate a student with six subjects would have a maximum of two hours of homework per night at the high school level. A student at the junior high level would have a maximum of one and a half hours of homework per night. The least amount of homework for both levels of students might be none at all since every teacher might chose not to assign homework on that day. Hence, an individual teacher needs to know what other teachers are assigning for homework, or a school might need a staggered system of homework for each course on specified days, or students might need to inform the teacher of heavy homework evenings. At the elementary level forty-five minutes of homework per night for grades one through three and sixty minutes of homework per night for grades four through six would be a maximum ("Information," 1981). However, some educators indicate that no homework should be given in grades one to three, and only classwork not finished during school time should be given in grades four to six. Homework should be regularly assigned. However, with the new national emphasis on homework, schools need a homework policy so that students are not overloaded with homework.

2. Homework should be clearly stated. (Foyle, 1984) If the teacher expects students to complete a certain assignment, writing the assignment on the board is appropriate. However, a clearly written homework instruction sheet which states the assignment is better. Students often miscopy assignments written on the board or even forget to write the assignment down. Students can write

their responses on the instruction sheet. Thus, the assignment is clearly stated and accurately placed in the hands of the student. Best of all, students do not hand in odd sized paper, papers with torn edges, or papers with confetti about ready to fall from the edges.

3. Homework should be regularly collected. (Shockley, 1964). Encouraging students to do voluntary homework does not always work, especially with less motivated students. Required homework must be collected or students will not complete the assignment. With the best of intentions, students often do not complete voluntary homework due to their other priorities. When homework is collected from students, the teacher shows the students that homework is important.

4. Homework should be promptly graded. By assigning a grade to the homework the teacher once again says that the homework is important. The teacher who records homework as completed but does not indicate a score on the homework may give the students the impression that the work was not important enough to read and grade. However, the whole assignment does not need to be read or graded. The teacher may select a portion of the assignment for grading. Spot checking of homework assignments produces similar student achievement results as complete checking of homework (Austin & Austin, 1974). Likewise, comments do not need to be made on each homework assignment or about each part of the homework assignment (Austin, 1976).

5. Homework should be promptly returned. Immediate feedback increases student achievement (Vitale & Hebbler, 1978; Cardelle & Corno, 1981; Ghandoura, 1982; Cardelle-Elawar, 1982). However, as every teacher knows, grading papers takes time. Homework can and should be returned during the next class period or day. Spot checking and limited comments will speed the grading of the homework and, hence, speed the returning of the papers. Upon returning homework papers, the teacher should comment upon items that were misconstrued or incorrectly completed. This process would take only moments away from the regular classroom presentation.

A HOMEWORK GUIDE

GUIDELINES	TYPES
Homework should be ● regularly assigned ● clearly stated ● regularly collected ● promptly graded ● promptly returned	■ Preparation ■ Practice ■ Extension ■ Creativity

Types of Homework

Variety is the key to a winning basketball game. If the same play is used over and over again, the opposition not only figures out what you are doing but will beat you in the process. The game would be boring to watch if the same play is used over and over again. Just like a basketball game, teachers can bring variety to homework assignments. Four basic types of homework can be assigned to students in order to provide that variety (Lee & Pruitt, 1979; LaConte, 1981).

● ***Preparation homework.*** One of the most common types of homework is preparation homework which is found in textbook reading courses. Preparation homework is getting ready *before* the classroom lesson. Usually students are assigned a section to read and questions to answer that cover the section. Then, the teacher covers the material during the classroom lesson. Preparation homework (getting ready) is commonly assigned in social studies and language arts classes.

● ***Practice homework.*** Another common type of homework is practice homework which is found in

skill-building courses. Practice homework is doing something again *after* the concept or skill was taught. The teacher covers a concept or skill during the classroom lesson. Students practice the concept or skill during class time or as homework in order to do the concept or skill once again. Practice homework (doing again) is often found in mathematics, typing, and shorthand.

● ***Extension homework.*** A less common form of homework is extension homework which is assigned in order to observe whether a student can transfer an idea or skill to a new or different situation. A teacher can assign extension homework which indicates whether or not a student can go beyond basic facts or skills. The mathematics basic skill of figuring the distance between two points on a line is incorporated into a story problem about travelling from one place to another. In typing, the basic skill of using certain typewriter keys is incorporated into a business letter. In social studies and language arts, students write a newspaper story about a coal miners' strike in 1894 or the General Motors' strike in 1984. Extension homework (going beyond) is often found in the high school newspaper or the elementary school "Dear Santa" letter.

● ***Creativity homework.*** Probably the least used form of homework is creativity homework which is found at all grade levels and subject areas. Creativity homework is putting together concepts and skills in new and different ways. Usually, creativity homework takes longer to complete than preparation, practice, or extension homework. At the elementary level, a student could build a kite which would incorporate reading about kites (history), building the kite (mathematics), painting the kite (art), explaining the process to the class (language arts), and flying the kite (physical education). At the high school level a research project lends itself to this type of homework. Students can be creative in topic choice and demonstration of the knowledge or skill. Creativity homework (putting together) is a way in which students can integrate many skills and concepts.

Foyle (1984) conducted a class room homework experiment in the tenth grade which utilized the above homework guide. Six intact American history classes (131 students) were divided into three groups: (1) preparation homework, (2) practice homework, and (3) no homework. Foyle used an analysis of covariance to hold student ability and previous course knowledge constant. The researcher found that (a) factual content homework increased the students' achievement when compared to the students who were not assigned homework, (b) either preparation homework or practice homework can be assigned to students since both types of homework raise students' achievement when compared to the students who were not assigned homework, and (c) females and males achieved the same regardless of the type of homework assigned to them.

Thus, teachers may choose from these four types of homework in order to provide variety in their game plan. Our homework research, homework experiment, and teaching experience substantiate that by using the five homework guidelines as starters and the four homework types as a game plan, teachers can win the game of student achievement. **R**

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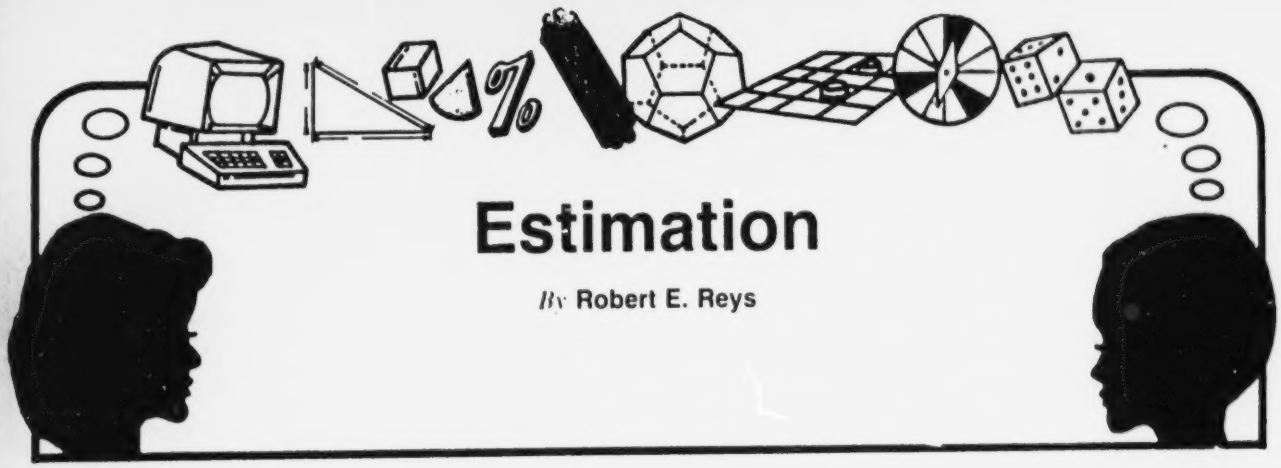
Estimation Resources



Estimation

By Robert E. Reys

Estimation and mathematical thinking go hand in hand. Both are multifaceted, and each involves many different processes. I feel that instruction in estimation provides a natural context not only to develop but also to practice many important thinking skills.

What Are Some Important Thinking Skills?

An examination of the articles in this issue of the *Arithmetic Teacher* shows that what constitutes mathematical thinking is subject to many different interpretations. Rather than trying to formulate an explicit definition of mathematical thinking, let's consider some of its characteristics. For example, in mathematical thinking, a person—

1. looks ahead in a problem and decides what type of answer is needed;
2. uses mental flexibility in working with different forms of numbers;
3. selects appropriate strategies;
4. recognizes that multiple solutions exist and is not afraid to abort one strategy in favor of another; and
5. checks for reasonableness of results.

Activity 1

Read each newspaper headline.
Decide whether the numbers represent *exact* data or *estimates*.

World Series Attendance: 325 000
369 Killed in Crash
Star Signs $500 000 Contract
Market drops 7.25 points
Big Win 189–180
2.5 Billion Sold

If these sound like characteristics of problem solving, that's because they are. Problem solving and mathematical thinking are inseparable. Even so, this list is not exhaustive, nor are the characteristics mutually exclusive. They do, however, highlight some key thinking processes that can be furthered by the teaching of estimation. The remainder of this article expands on these skills and provides a few specific illustrations of how estimation can help better establish these skills.

Looking ahead

Many times we need an exact answer to a problem; however, many more times an estimate will do just as well. Since estimates are often easier and quicker to produce, we should use them. Children need to realize that today's world is bombarded with numerical data, some exact and others estimates. It is helpful and often very important to know which is which. The newspaper headlines reported in activity 1 can be used to help students better discriminate between estimates and exact answers. As these headlines are discussed, not everyone will agree on how to classify each of them, but this experience will encourage children to think about and defend, or at least explain, their interpretations.

Of course, exact answers are needed many times (e.g., The station is 102 on the FM dial; How much is our electric bill this month?). Many times estimates are reported (e.g., The station is around 100 on the FM dial; About how much will our electric bill

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be next month?). Questions such as those in activity 2 provide opportunities to sharpen recognition as to whether exact answers or estimates are needed. Looking ahead and deciding the type of answer needed for a problem is an important part of mathematical thinking. It not only sharpens students' awareness of the power and usefulness of estimation but it often provides direction for the eventual solution of the problem.

Mental flexibility with numbers

Recognizing what type of answer is needed is a first step toward mental flexibility, and a good number sense is the next step. Consider, for example, the following problem:

Estimate: $\frac{7}{8} + 1\frac{12}{11}$

Students with a good number sense recognize that these fractions are near 1 and 2, respectively. Such thinking turns a potentially long, messy algorithmic solution into a quick response of "almost 3." Bright students frequently exhibit this type of thinking on their own, but the majority of students need careful instruction to develop number sense and flexibility. Activity 3 illustrates one way flexibility might be strengthened.

It is essential that instruction be designed to offer a balance of activities to increase students' flexibility with numbers. These activities include the linking of different numbers, such as translating "Sale—35 percent off" to "1/3 off," or recognizing that 1/10 is not only close to 0.0897 but much easier to use in producing an estimate. Mental flexibility is also essential for estimation and represents an important element of mathematical thinking.

Selecting a strategy

Many different strategies are useful in producing estimates. Although rounding has been the predominant estimation strategy taught, other powerful and useful strategies, such as front-end, averaging, and compatible numbers, are available (Reys and Reys 1983; Trafton 1976). Each of these strategies is illustrated in figure 1.

Research has demonstrated that

Activity 2

Is an estimate sufficient when—

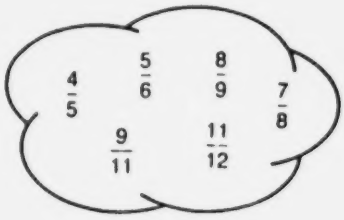
- the accountant figures out how much money was made on ticket sales?
- the newspaper reports how many people attended the game?

Is an estimate sufficient when—

- the waitress figures 5 percent tax?
- the waitress finds the total?
- the customer figures a 15 percent tip?
- the customer checks the bill?

Activity 3

These fractions are all less than 1.



Suppose two of them are added:

$\frac{8}{9} + \frac{9}{11}$

Together, it's *less than* 2 pies.

The sum must be . . .

less than 2 | exactly 2 | more than 2

Suppose three of these fractions are added. The sum must be

_____.

Think why.

Try these!

1. $\frac{7}{8} + \frac{4}{5}$ less than / more than _____.
2. $\frac{1}{3} + \frac{7}{15}$ less than / more than _____.
3. $\frac{5}{9} + \frac{7}{13} + \frac{2}{3}$ less than / more than _____.
4. $\frac{4}{5} + \frac{11}{12} + \frac{8}{9}$ less than / more than _____.
5. $\frac{7}{9} + \frac{10}{11}$ is more than _____ but less than _____.
6. $\frac{4}{9} + \frac{6}{13} + \frac{9}{19}$ is more than _____ but less than _____.

Fig. 1

Front-End

Estimate the total of this grocery ticket.

AGr	4.19
Pr	.86
Pr	1.39
AGr	.29
Mt	2.14
Tax	.23
	.10

4 + 1 + 2

1 **Front-end** —The dollars total $7.

2 **Adjust** —The cents are about $2.

19 and 86 are a dollar. The rest make a dollar.

3 **Estimate** — $7 + $2 = $9.

Averaging

Estimate the total attendance.

Mon.	72 250
Tues.	63 891
Wed.	67 490
Thurs.	73 180
Fri.	74 918
Sat.	68 490

The figures all cluster around 70 000, so about 70 000 people attended each day.

6 × 70 000 = 420 000

Averaging estimate: 420 000

Compatible Numbers

Estimate the monthly payments.

Financed cost: $15 629

Payments: 48 months

The problem is easier to think of as $50\overline{)15\,000}$.

Compatible number estimate: $300

when these and other strategies are taught, students' performance increases significantly (Reys et al. 1984). It is essential that a variety of estimation strategies be taught systematically within mathematics programs. Research has also shown that unless specific strategies are taught, few students will develop them on their own. Thus as their repertoire of estimation strategies increases through instruction and practice, students become more aware of the choices that can be made. An analysis of the conditions of the problem, along with students' own preferences and styles, will lead to a solution process that matches a strategy to a problem. These analytic, judgmental, and decision-making processes are at the heart of mathematical thinking.

Multiple solution strategies

The adage "there is more than one way to skin a cat" applies particularly well to estimation. Often an estimate can be produced in many different ways, as shown by the problem in figure 2. Each of these strategies is different, but any of them would be appropriate to use. The discussion and sharing of different approaches in the classroom will not only decrease tunnel vision (the notion that things can only be done one way) but also lessen the "one right answer" syndrome.

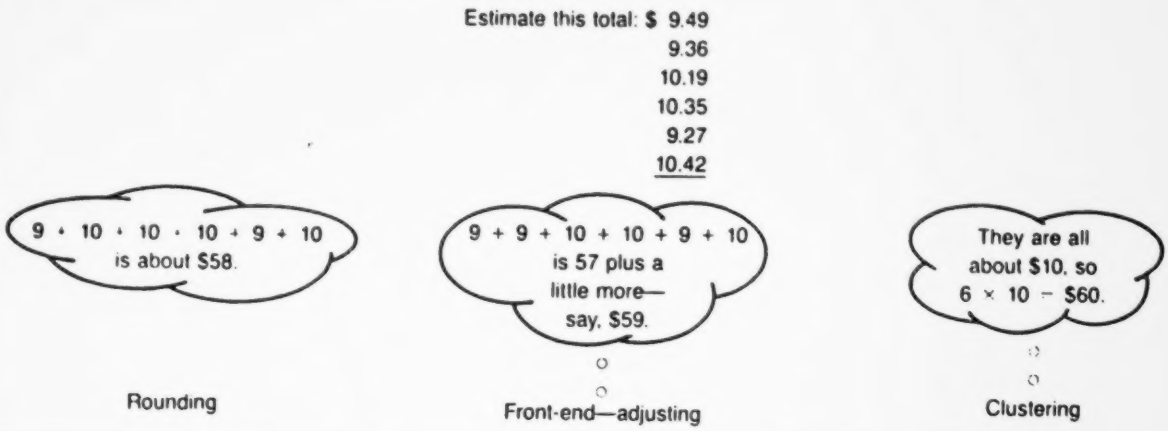
Along with the acquisition of a variety of estimation strategies should come the recognition that several of them can be applied to the same problem. If one strategy becomes messy or appears unproductive, try another. Estimation should be done quickly, and if it becomes laborious, a new strategy should be applied. This flexibility of thinking and willingness to try another approach helps build students' self-confidence. It is not only an important but also an essential part of developing mathematical thinking.

Checks for reasonableness

Is an answer reasonable? That question should be asked every time an answer is produced. Each of us has our own horror stories about unreasonable answers. They happen both in and out of school. Often these unreasonable answers are associated with "way out" results, but not always. An answer can be close but still be unreasonable. One vivid illustration of this phenomenon actually happened to me while checking out at a discount store. Tennis balls were on sale for $1.99 a can. I purchased exactly 15 cans of balls and no other items. The subtotal on the register (before tax) was $31.94. Was this answer unreasonable? It was to me but not to the cashier. Perhaps the most disconcerting thing was how the cashier handled this conflict. Rather than "think" about the numbers involved (i.e., 15 cans at $1.99 each), the cashier chose to ring up all the items again. This algorithmic behavior and total dependence on technology of a person who had "successfully" completed high school was disappointing.

How can we tell if an answer is reasonable? Generally two types of

Fig. 2


Estimate this total: $ 9.49
9.36
10.19
10.35
9.27
10.42
9 + 10 + 10 + 10 + 9 + 10 is about $58.
9 + 9 + 10 + 10 + 9 + 10 is 57 plus a little more— say, $59.
They are all about $10, so 6 × 10 = $60.
Rounding
Front-end—adjusting
Clustering

criteria are used. One is related to the context of the problem. Activity 4 provides opportunities to check whether the answers are realistic for the given situation. A second criterion often involves estimation and is related to the manipulation of numbers. Activity 5 illustrates one way to practice and develop this type of thinking. Encouraging regular but not perfunctory checks for reasonable results should be a central focus of mathematics instruction. Only through systematic instruction and consistent encouragement will thoughtful checks for reasonable answers become an integral part of students' mathematical thinking.

Conclusion

The process of estimation is not only compatible with mathematical thinking but also conducive to it. Often we have a minute or two in class before recess, lunch, or an assembly; an available pool of estimation "quickies" can help us take advantage of those minutes. See figure 3. Although these problems beg to be done quickly, they offer practice in estimation and mathematical thinking simultaneously. Many other interesting and challenging problems of this type are available, for example, Reys and Reys (1983).

Every component of estimation—deciding on the type of answer required, choosing and carrying out appropriate strategies, and checking the

Activity 4

Choose the most reasonable number:

A new baseball glove costs _____ .

$1.40 $14.00 $140.00

The number of students who attend Washington School is _____ .

5 50 500

The Gonzales family of four went to Speedy Burger for lunch. They spent about _____ .

$8.00 $48.00 $80.00

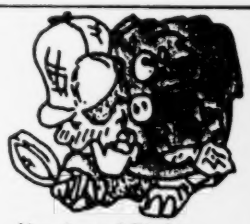
Fill in a reasonable amount:

Our school has about _____ students.

The average number of people in a family is _____ .

About _____ people live in the United States.

Activity 5



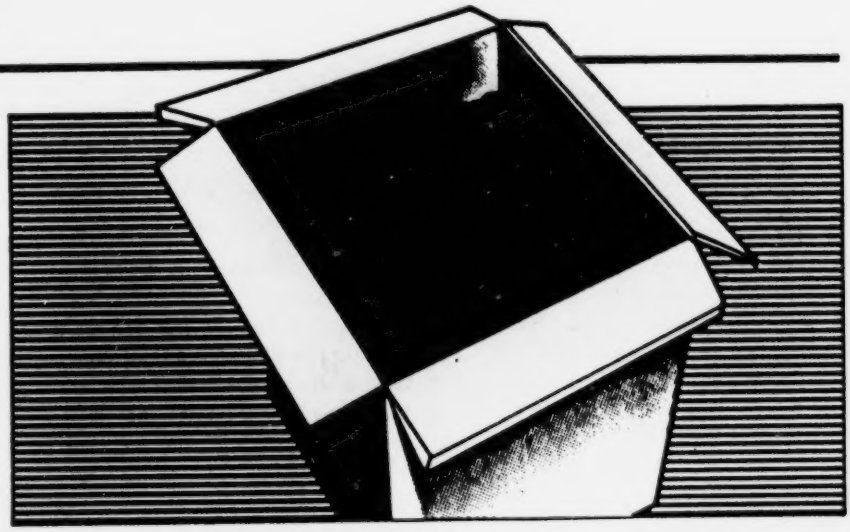
How Many Digits?
Study each problem. *Don't* estimate the answer.
Decide *how many digits* each answer should contain.

Problem	Number of Digits
1. 134 + 689	_____
2. 134 + 989	_____
3. 12 × 234	_____
4. 9 × 38 × 19	_____
5. 1764 − 783	_____
6. $4\overline{)2345}$	_____
7. $14\overline{)2345}$	_____

Let's Do It

Estimate How Much

By **John Van de Walle,** *Virginia Commonwealth University, Richmond, VA 23284*
and **Charles S. Thompson,** *University of Louisville, Louisville, KY 40208*



Let's Do It—activity approaches to teaching and learning mathematics in the primary grades, with extensions through grade six.

Young children need a wide variety of experience with the related concepts of number, quantity, and measure. Whereas children's initial activities with numbers focus on the counting of objects, later experiences should also require them to observe sets of objects, distances, and areas and then relate to these observations appropriate numeric estimates of the quantities or measures involved. Such estimating experiences provide a broad, practical context for continued development of children's concepts of number. Further, the ability to estimate or approximate numbers or quantities is a basic problem-solving skill.

The activities presented here are designed to help young children learn the basic notion "about how much."

The Meaning of About

The first and most difficult hurdle in helping young children respond with approximated or estimated answers is to convince them that numerical responses that are close to the exact answer are good and that the exact answer is not required. The following is one good way to begin with the notion of *about*.

Collect about thirty small counters that will show up nicely on an overhead projector. Macaroni, buttons, centimeter cubes, toothpicks, paper clips, or bottle caps are suitable materials. Start with fewer than ten counters. Place them randomly on the overhead projector's stage but with the light turned off. Be sure the children are all looking at the dark screen. Now flip the light on for only a second or two. Have the children write down about how many things they saw. Then turn on the overhead light and have the children count the objects together. Discuss what a "close" answer or a good "about" answer might be. For three objects, two, three, and four are all good estimates. For eight objects, estimates of five or eleven are at least in the ball park. Better estimates are the numbers from six through ten, and all should be considered as equally good. The reason for having children write their responses is to prevent them from following the lead of the first or loudest response.

In this type of activity, one of your major goals is to help children accept the idea that "close but not exact" is quite appropriate when the opportunity to count is not provided. At first, some children will not be willing to respond at all, since they cannot believe that you are not expecting a single "correct" response. Their other experiences in mathematics have taught them that if an answer of four is correct, an answer of three or six is wrong. With this in mind they are unwilling to approximate. Other children will understand you to mean "guess how many" and will give wild or seemingly unrelated responses. These children understand "guess" in the random sense of the word as used in "guess my number" games. Therefore, it is important to discuss the idea of a range of good approximations and to avoid giving any special credit to children whose approximation happens to be exact.

This discussion of estimation, or "about how much," will need to be repeated frequently, certainly every time you begin a new "about" activity. Eventually the children will understand what you mean when you say, "About how many do you think there are?" or "About how long do you think this is?" The concept of

"about," however, does not come easily. Children in the upper grades will try to figure exact answers to such questions and then add or subtract a little, or round off the result, because their notion is that "about" necessarily means "not exact."

About How Many

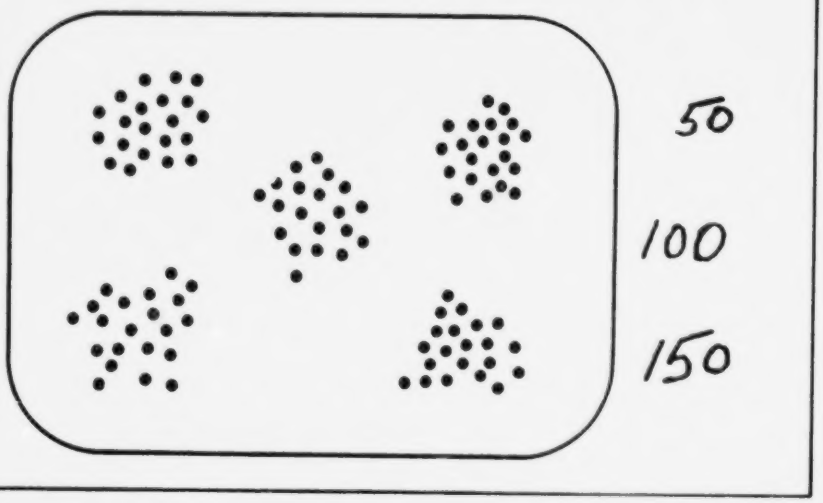
As soon as the notion of "about" is understood, use larger amounts in estimating activities. Eventually some kindergarten children can work well with amounts in the teens. However, work with the numbers between 4 and 12 is most appropriate. Children in the first through third grades can work with amounts from 20 to perhaps 1000.

Another important variation of this basic activity—placing counters on the overhead projector—is to change the type of counters being used. Very small counters, such as pinto beans, provide quite a different visual image than do larger counters, such as bottle caps. Toothpicks or small sections of plastic straws provide a still different look. By propping up a box lid in front of the overhead projector, you can change the type of counters for each new set presented without the counters being seen by the class. In this way the children will not know either the type or size of counters to expect.

The overhead-projector activity can be modified for small groups or even pairs of children working independently by placing the counters under a box top or piece of tagboard. Instead of flashing the overhead lamp, remove the box lid for a second or two and then replace it to hide the counters.

Estimates of more meaningful quantities can be made by playing "how many in the room?" Have all children close their eyes. Select a set of something in the room, for example, the papers on a bulletin board. Ask, "About how many spelling papers are on the green bulletin board?" Children must write their estimates without looking. Discuss the results as before. Other items in the room to estimate might include windowpanes, pictures, plants, ceiling tiles, books on your desk, crayons in the crayon box, stars on a progress chart, leaves on a particular plant, and so forth.

Fig. 1 Multiple choice is used for estimates of large amounts.



A next step in helping children learn to estimate the size of a set is to have them mentally combine several easily countable, smaller subsets that compose a larger set. Return to the basic counter activity but, without mentioning it ahead of time, set out the counters in subsets of nearly equal sizes. For example, place the counters in three sets of four counters each. After completing several examples of obvious groupings, ask a few of the better estimators if they saw anything that helped them with their "guesses." Wait until some of the children have caught on before explaining the idea to the entire class. The students will frequently be able to verbalize a new idea to their peers even better than you can. For older children you can even arrange the counters in equal piles that cannot possibly be counted in the allotted time but that can be estimated. For example, put from twelve to fifteen in each pile. Then allow students some time with pencil and paper to calculate their estimates. Even second graders can do this by using addition (or skip counting when smaller groups are used).

As the quantities that you are using get larger, the probability of making a close estimate becomes more remote. To avoid any discouragement, provide a multiple-choice format for each estimating task. As shown in figure 1, simply write three or four possible estimates on the chalkboard. Always present the choices in numerical order with about the same difference between successive choices. Students then select the amount that they think is closest to the actual amount shown. In the beginning make the difference between the choices quite large. For a set of 54 you might offer 50, 150, and 250 as choices. Later, choices of 50, 100, and 250 might be more appropriate. (It will be necessary to precount your sets so that you can select good choices.) This same multiple-choice approach may also be helpful in the very beginning for students who are reluctant to guess.

As you vary and expand your numeric estimating activities, continue to vary the objects used as counters. Try putting them in a large glass jar instead of on the overhead projector's stage. Put slightly larger items in a paper or plastic bag and squeeze out the loose space so the volume of the counters is clearly visible. If using a paper bag, tell the class what is in the bag and show one or two of the items that are inside. Pencils, walnuts, crayons, spools, blocks, wrapped candy, and jar lids are examples of the wide variety of materials that could be used.

Plan an "about how many" activity with one to three estimates for each day. Since it will only take a few

Fig 2 Rods used to measure a book

minutes to do an activity, you do not have to wait for math time. Use them as fillers or change-of-pace activities. Always have a few sets bagged and ready to go.

About How Long or How Far

Once the notion of "about" is clear to your students, you can apply it to measures of length. This adds the concept of linear measurement to the estimating activity.

Before beginning the estimating activities, introduce the idea of measuring length using various units. For example, measure a line made from masking tape placed on the floor by using shoe lengths as the unit. (Keep the distances small so that the measurements involved will be less than ten.) Measure desk tops, children's heights, doorways, and bookshelves. As units use hand spans, index cards, plastic straws, chalkboard erasers, building blocks, pencils, or any object that has an obvious dimension of length. Measure small items with small units. Centimeter cubes, paper clips, bottle caps, Cuisenaire rods, and Unifix cubes are just a few of the objects that can be used to measure the length of a book or a pencil or the height of a vase.

Initially, kindergarten and first-grade children need to measure lengths by actually placing units end to end, as shown in figure 2. Only later can they use a single object as the unit and move it along the length being measured.

When children have become reasonably familiar with what it means to measure something with a specified unit, you can begin the "about" activities. Present a length and a unit and ask, for example, "About how many erasers long do you think this rope is?" Children write their estimates, the length is measured, and the results discussed as described with the "counter" activities in the previous section.

Initially, use the same unit for several length-estimating activities. Help children see that a new object to be measured is shorter, longer, or nearly the same length (or height) as a recently measured object. In this way you are beginning to introduce "benchmarks," or reference points, for the children to use in their estimates. Develop several benchmarks that will be relatively stable and useful not only in your classroom but in other places as well. The heights of doorknobs and doorways are roughly the same in all buildings. A child's arm span, hand span, shoe length, and pace will remain nearly the same over a year.

After a week or so of estimating and measuring various lengths with one unit, return to the same lengths and measure them with a different unit that is clearly longer or shorter than the first one. Suppose, for example, that you have measured several objects with a cardboard tracing of the principal's shoe. Measure the same objects with a chalkboard eraser, which may be about half as long. Do not be surprised to find that children will *think* the objects will have smaller measures when the smaller unit is used. The inverse relationship between the size of a unit and a measurement using that unit will often not be clear to most children below the third grade.

This inverse relationship should be demonstrated with reference points, or benchmarks, that the children have established. Let's suppose that a student's desk has been selected as a reference point. Note that, for example, it is eight hands but only four new pencils across the desk. Discuss these results with the children, focusing on the idea that "the longer the unit, the fewer of them it takes to be as long as the item being measured."

The most obvious extension of these activities is to use standard units of length, such as centimeters, decimeters, and meters, instead of the informal units suggested so far. But do not be too anxious to move to standard units and rulers, as these can be abstract and difficult for children if sufficient introduction to them has not been made. (See "Let's Do It: Learning about Rulers and Measuring" in the April 1985 issue of the *Arithmetic Teacher*.) However, metric measuring activities will be much more interesting to your students in the context of these estimating activities rather than in activities that simply involve measuring items and recording the results. Furthermore, benchmarks become more stable when related to metric units and, in fact, can become models for the metric units themselves. For example, find objects that are about one meter wide or tall and help children use these reference points to determine the lengths or heights of other objects.

Another variation of these activities is to move them outdoors or to the gym, where longer distances can be measured. The measuring units must be very large to keep the numbers within a reasonable range. A meterstick or a meter rope (a length of rope cut to one meter) will work adequately for some distances. A five- or ten-meter rope will be better for longer distances, such as the length of the playground, a building, or a fence.

Curved or circular distances are the most difficult to estimate. Begin estimating "about how far around" with

objects in your classroom after children have become reasonably skillful with straight-line distances. Things to measure around include flower pots; pencil holders; wastebaskets; children's waists, heads, and arms; balls; and round table tops. Of course, "around" does not apply only to round things. You can also measure around desk tops, books, chairs, and television sets.

Perhaps the easiest way to measure around an object is to wrap it with string or yarn and then measure the resulting length of string. This indirect method may be obscure to some children, so be sure to help them understand what they are doing.

About How Big

Another possible application of the "about" activities is in the measuring of areas. The areas of small regions, such as book covers, lunch trays, pictures, pieces of paper, or the bottom of a wastebasket, can all be measured roughly by covering them with precut squares of corrugated cardboard or tagboard. Cardboard can easily be cut into squares with a paper cutter. A collection of thirty or forty 10 cm ×, 10 cm squares will work nicely for most small regions that you may want to measure in your classroom.

To measure a region, simply cover it with squares as well as you can. In most cases the squares will not fit the region exactly. Discuss with the students the fact that in some places, the squares overlap the region, and in others the region extends beyond the squares. Both possibilities are shown in figure 3. Try to balance them so that counting the squares will indicate about how big the region is. No attempt should be made to develop formulas for finding areas.

Bigger regions can be measured similarly by using larger squares. Sheets of tagboard can be cut into squares that are about 50 cm on a side. Sheets of newspaper can also be cut into large squares that can be used to measure such regions as table tops, rugs, hopscotch areas, or regions outlined on the floor with chalk or masking tape.

Fig. 3 Cardboard squares measure the area of an irregular region.

The children will find it more difficult to estimate area, but this should not discourage you from conducting these "about how big" activities, even with the youngest children. Besides the numeric value gained from estimating, counting, and comparing, the concept that area is a measure of the covering of a region will be developed in an informal and interesting way.

Perhaps the most interesting aspect of the "about how big" activities is to contrast regions having quite different shapes. It is quite surprising that long skinny shapes can have areas that are smaller than those of square or nearly square shapes that appear smaller. For example, a 5-by-5 square is five squares larger than a 1-by-20 rectangle. By drawing contrasting shapes like these on the floor, you can have fun discussing the results with your class.

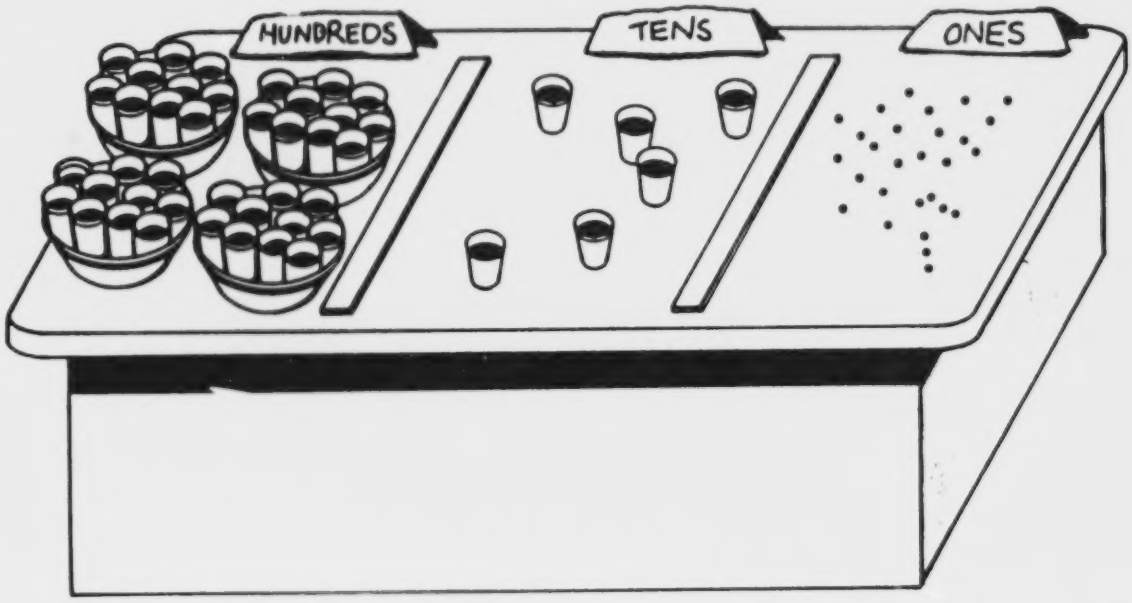
About How Many with Very Large Amounts

Beginning late in the second grade and continuing through elementary school, children are exposed to numbers in the hundreds, thousands, and even millions, but they have almost no realistic, firsthand experiences to associate with those numbers. "About how many" activities provide an interesting opportunity to expose children to large-number concepts.

Start with sets that are large but still quite countable. A jar of lima beans, a bag of candy, or a shoe box filled with small stones are examples of sets that are easy to collect. Foam packing chips, plastic centimeter cubes, or a stack of writing paper are sets that you may already have in your classroom. Cut some newspaper into pieces the size of a dollar bill and create a garbage bag full of "money." These and other countable collections can each be used several times by changing the amounts or by placing the items in containers of different shapes. Beans will look quite different in a jar than in a shallow box.

The children should be given several days to examine these sets and discuss their estimates of "about how many" with their classmates. Have children write their names and estimates on slips of paper and deposit them in a box. When all children have made an estimate, count the items in the collection as a class or group project. Let each child count a portion of the set by counting out several piles of ten. With all children observing, combine these piles into sets of a hundred and, if necessary, combine sets of ten hundreds into sets of a

Fig. 4 Counting out small objects by ones, tens, and hundreds



thousand. If the sets of ten are placed in small paper cups, the combining and grouping can even be done on a large place-value chart drawn on a large table or on the floor, as shown in figure 4. Larger containers can be used for the hundreds and thousands and placed appropriately on the chart. In this way valuable place-value concepts are developed along with the large-number concepts.

Once the final count has been made, the children's estimates can be taken from the box and listed (without names) in numerical order on the chalkboard. A discussion of what constitutes a good estimate for a particular number adds another valuable dimension to the activity.

For intermediate-grade children, several other types of large sets are accessible. Patterned sets appear everywhere: holes in a chain-link fence, bricks in a wall or sidewalk, floor tiles in a hallway, holes in the edges of computer paper, squares in a ream of graph paper, or holes in a window screen. Generally these can be counted, or at least carefully estimated, by counting rows, columns, or other appropriate subsets and using multiplication to determine the totals.

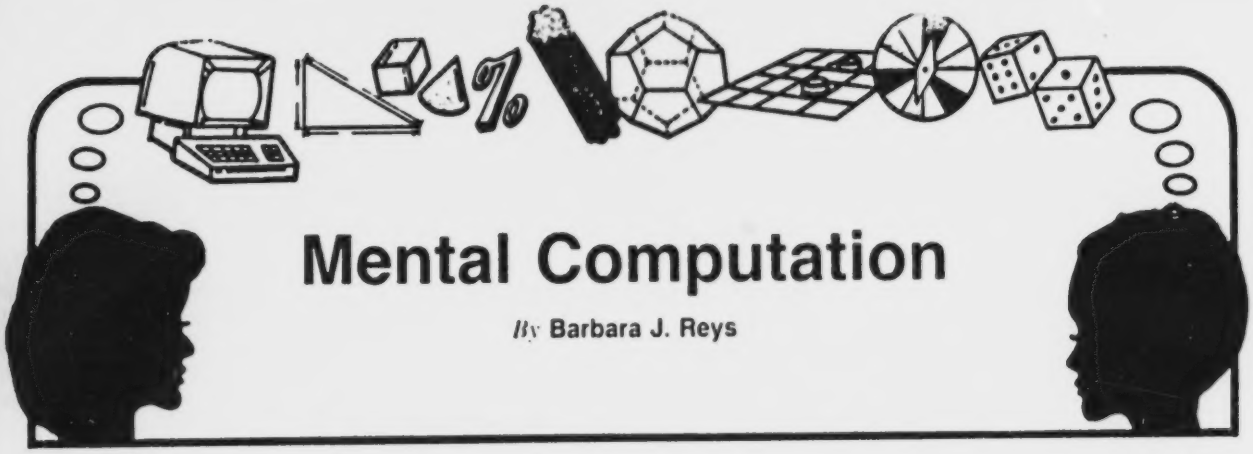
Certain "uncountable" sets provide another challenge. Grains in a cup of sand, needles on a small pine tree, blades of grass in a well-defined region of a lawn, letters on the pages of a book, and drops in a gallon of water are all examples of sets that can reasonably be estimated. In each case a suitable counting method consists of finding one or two representative subsets that can actually be counted (e.g., a square decimeter of grass, an eighth of a cup of water) and counting the objects in that subset. Then determine how many such subsets are in the total set, a task that is usually a bit easier than actually trying to count the entire set.

A third type of large set is imaginary and can be estimated by calculating. The number of pennies it would take to cover the floor, the number of steps an ant would take to walk around the building, how many seconds it would take to count to a million, or the number of candy bars it would take to make a ton are just a few of the ideas you and your students can have fun exploring.

Conclusion

The "about" activities we have presented can easily be adopted for spare moments during the day. They also can be used as the basis for a unit on number concepts, place value, or measurement. They are fun and easy to do.

Moreover, these activities help young children to develop further their concepts of number and quantity, their understanding of the measuring process, the use of comparison in making estimates, and the general meaning of estimation. Such results are worth the small effort required to help children estimate "about how much." ■



Mental Computation

By **Barbara J. Reys**

The ability to compute mentally, that is, to calculate exact numerical answers without the aid of any calculating or recording device, varies tremendously among individuals. A quick survey of any elementary or secondary school classroom will document that some students perform mental computation quickly and accurately, whereas others are greatly hampered on even simple arithmetic problems by the withdrawal of paper and pencil. Occasionally history reports a case of an individual who has developed extraordinary skill in mental computation. These individuals have both fascinated us and made us curious about the development of such skill. One such example was Zerah Colburn. Born in Vermont in 1804, the son of a farmer, by the age of eight he was touring America and England displaying his exceptional mental calculating ability. He was able to give instantly the product of any two four-digit numbers. Asked to raise 8 to the sixteenth power, he gave the correct answer of

281 474 976 710 656

in a few seconds without the aid of any recording device (Eves 1969).

How were Colburn and others like him able to compute mentally at such a high level? The methods used by expert mental calculators vary, and many of their processes involve properties of numbers, highly efficient short-term memory, memorization of extended addition and multiplication tables, and advanced arithmetic and algebraic techniques.

Obviously this kind of skill is not to be expected of all schoolchildren. Yet it is becoming more obvious that students need and can greatly benefit from some skill in mentally computing arithmetic problems. Students should not be limited to written computation but should take advantage of whatever type of computation is most efficient for a problem—written, mental, or mechanical (calculators). An important area of instruction that needs attention is helping students decide among these methods for specific arithmetical situations.

Although instruction and practice in mental computation can improve skill in written computation, their major value appears to be in helping students understand the flexibility and power of our number system. The "thinking" activities that are critical to mental computation give students a "sense" for computational routines and a "feel" for numbers.

The recent surge of interest in establishing mental computation in the mathematics curriculum is not a new phenomenon; mental computation in the school curriculum has a long and sporadic history. As early as the middle of the nineteenth century, mental arithmetic was an integral part of mathematics instruction. Mental arithmetic was used as a form of exercise to improve general mental discipline, and it often involved intense oral drill on problems requiring several steps to complete (Buchanan 1978). The following examples of "mental" problems are taken from *Higher Arithmetic*, the 1896 textbook for upper elementary school students by Walsh; they help illustrate the extreme to which skill in mental arithmetic was expected and practiced. Such problems are included in each section of the text as a regular feature.

> A farmer makes 675 gallons of cider. He has but 12 barrels, each of 45 gallons' capacity, to store it in. How many more such barrels does he need?
>
> What profit is made on an article bought for $175, less 12%, and sold for $200?
>
> Adam lost $22. He then found 1/4 as much as he had remaining, and then he had 1/3 as much as he had at first. How much had he at first?

By the early 1900s this emphasis on intense mental drill was followed by a counterreaction. The theory of mental discipline fell into such disrepute that it was difficult for any form of mental arithmetic to maintain a place in the curriculum. Up until this point, few educators had conceived of a curriculum placing appropriate emphasis on both mental and written computation. One such far-sighted educator, however, did propose a balanced curriculum (Suzzallo 1911):

> It is altogether probable that many simple calculations or analysis can be done 'silently' (mentally) from the beginning, that others require visual demonstration, but once mastered can thereafter be done without visual aids, that still others will always have to be performed, partially at least, with some written work. It is purely a matter for concrete judgement in each special case, but the existing practice scarcely recognizes this truth. The result is that many problems are arbitrarily done in one way, and it is too frequently the uneconomical and inefficient way that is used.

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The Easy Way

Find each answer *mentally*. Study the problem before you start. Look for an easy way to do the problem.

1. 5 + 7 + 3 + 5 + 8 + 6 + 2 =

2. 37 + 35 + 63 + 75 =

3. 2 × 3 × 7 × 5 =

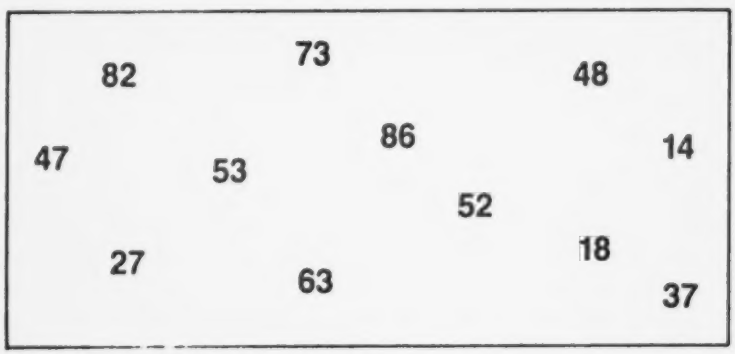
4. 4 × 17 × 250 =

5. 5 × 25 × 4 × 6 =

6. 8 × 19 × 14 × 10 × 0 × 2 × 1 =

Make 100

You have one minute to find as many pairs of numbers that add to 100 as you can. Use only these numbers.



Noticing Nines

All these problems involve numbers ending in 9. Study the example to learn an easy way to add such numbers mentally.

Now try these:

1. 56 + 29

2. 48 + 79

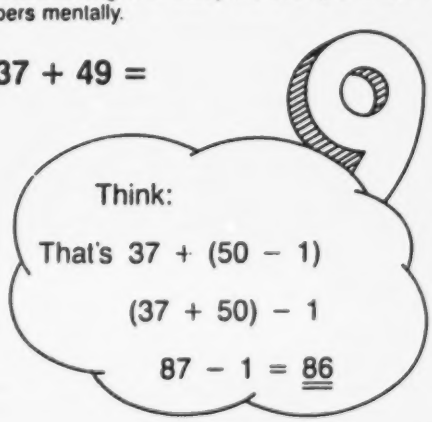
3. 67 + 19

4. 143 + 39

5. 34 + 99

6. 238 + 199

37 + 49 =



Between 1930 and 1960, concern again emerged that pupils were becoming too dependent on paper-pencil algorithms and that more practice in mental arithmetic was needed. The rationale for this revival centered around a pragmatic issue—mental arithmetic was more useful outside the classroom than was paper-pencil computation. Therefore students should experience arithmetic situations that more closely resembled those to be encountered in daily life.

The present decade has seen a revitalization of interest in mental computation. In part this interest has been spurred by a growing realization that many students apply written algorithms mechanically, with little sense as to why, how, or what they are doing.

What are the benefits of mental computation?

In the past, the inclusion of mental arithmetic in the school curriculum was justified on the basis of social utility or the theory of mental discipline. Current efforts seem to be more broadly based than efforts in the past, as evidenced by the dialogue of professional educators about the benefits of mental computation in today's curriculum. In particular, three practical and pedagogically sound reasons have emerged:

1. *Efficiency*. Many everyday uses of mathematics involve nonwritten arithmetic. Cash registers, electronic scanners, and automatic changers are used in grocery stores and restaurants. Computers and calculators are used in banks and other businesses to process numerical data. Tables are used in movie theaters to determine ticket prices. In short, our society takes advantage of the most efficient method of calculation for the situation at hand. At the same time, our present school mathematics curriculum stresses written algorithms for calculations. It is discouraging to observe a third-grade student resort to paper and pencil when asked to add 200 and 300. Yet this mechanical application of paper-and-pencil algorithms is all too common.

Some mental arithmetic and estimation skills are essential for checking possibly erroneous entries in computing devices. Many real-life problems presented to us can be handled by simple application of a mental algorithm. Although paper and pencil and calculators are not always conveniently at hand, our mind's calculator is always with us.

2. *Enriching the existing curriculum.* The inclusion of mental-arithmetic development offers teachers a vehicle to review and practice the computational skills that many junior high school students are experiencing for the fourth and fifth time. Mental computation can challenge even the brightest students yet includes important basic content that can be mastered by low-ability and average students as well. I have found that discussion and practice devoted to the mental computation of such problems as 3/4 of 240, 0.15 × 12, 25% of 840, and 20% of 16 help nurture a greater understanding of important concepts about fractions, decimals, and percentages than is achieved when paper-and-pencil algorithms are stressed.

3. *Developing mathematical reasoning.* Mental arithmetic combines a variety of skills—knowledge and application of various algorithms, flexibility in thinking, efficient use of the properties of numbers, an understanding of our base-ten number system, and the use of alternate forms of numbers. Nurturing these skills not only brings about skill in mental arithmetic but also produces considerable transfer to other important areas of the mathematics curriculum, including problem solving and estimation.

How are thinking skills enhanced?

A closer look at the cognitive thought processes used to compute mentally reveals some specific benefits of teaching mental computation. Several examples will help highlight these benefits.

Mental computation promotes an understanding of the base-ten number system as well as of basic number properties. It also encourages the use

Chaining

(To the teacher: Orally present each problem below, reading each slowly only once.)

1. 850 + 250 + 600 =
2. 475 + 50 + 175 + 550 =
3. 350 + 100 + 275 =
4. 875 + 175 + 450 + 2000 =
5. 550 + 425 + 150 =
6. 275 + 350 + 1000 + 550 =

How Many Digits?

For each problem, decide how many digits the answer should contain. You do not have to find the answer.

1. 236 + 684 =
2. 4876 + 6943 =
3. 1146 − 293 =
4. 14 263 − 5453 =
5. 27 × 5 =
6. 24 × 986 =
7. $4\overline{)7632}$
8. $34\overline{)28\,762}$

Use What You Know

1. If 47 × 5 is 235, what is 47 × 50? What is 470 × 500?
2. If 320 × 6 is 1920, what is 32 × 60? What is 3200 × 6?
3. If 49 + 15 is 64, what is 490 + 150? What is 49 000 + 15 000?
4. If 1413 ÷ 27 is 53, what is 1 413 000 ÷ 27 000?

and invention of alternate computational algorithms, for example:

547 + 199	"That's 547 + 200, or 747 minus 1, or 746."
$14 \times 3\frac{1}{2}$	"That's (14 × 3) + (14 × 1/2), or 42 + 7, or 49."

Mental computation encourages thoughtful inspection of a problem prior to the application of an algorithm.

5 × 27 × 20 × 3	"The easiest way to do this is to rearrange the factors. . . . 5 × 20 is 100, 3 × 27 is 81, so the product is 8100."

Mental computation rewards flexibility in dealing with various forms of numbers. The ability to translate a problem to a more mentally manageable form is a key ingredient of mental computation. Students benefit from searching for alternate forms of numbers that make their work more efficient.

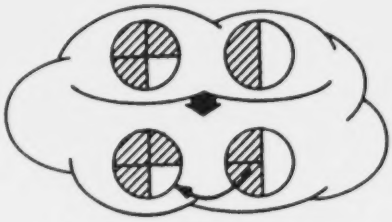
0.25 × 840	"That's the same as 1/4 of 840, or 840/4, or 210."
420 × 8	"That's 420 × 2 × 2 × 2, or 420 doubled 3 times . . . 840, 1680, 3360."

Mental computation nurtures the development of a keen number sense. Because algorithms are created and used, an important component of mental computation is the looking-back step to see if the answer makes sense.

60 × 15	"That's halfway between 60 × 10 and 60 × 20, or between 600 and 1200. . . . 900."
2/3 of 90	"That's the same as twice 1/3 of 90, or twice 30 . . . . 60, which makes sense because it should be less than 90."

Mental arithmetic utilizes visual thinking skills. For example, the use of pictorial models can often remove or lessen the computational burden.

3/4 + 1/2	"I can picture that . . . . it's 1 1/4."



Each of the benefits listed previously is in the spirit not only of establishing good thinking skills but also of encouraging students to think about mathematics meaningfully.

The mental computation curriculum?

Although mental computation is an important skill and one that has much payoff in other areas of the curriculum, it is not a topic that can be neatly fit into a certain grade level or sequence of the curriculum. However, certain types of experiences will help students at various levels become more sensitive to, and proficient in, mental computation. Obviously, a major portion of this curriculum in mental computation must focus on mental algorithms for certain operations and types of numbers (whole numbers, fractions, decimals, and percentages). However, other experiences will complement this instruction as well as provide other necessary components (number sensitivity, number-property review, and problem-solving skills). Several examples of these activities are highlighted throughout this article.

Whatever the specific curriculum, it should have a variety of instructional components:

- Development of, and proficiency with, efficient mental algorithms
- Attention to place value (especially how certain mathematical operations affect the place value of an answer)
- Inclusion of, and a healthy respect for, the creation and use of alternative algorithms
- Emphasis on thinking about an efficient solution strategy before attacking the problem
- Attention to various types of numbers, including not only whole numbers but fractions, decimals, and percentages as well
- Attention to oral mental computation as well as to visual presentation of problems

Mental arithmetic is an exciting and rewarding topic to introduce and develop with students. Growth occurs rapidly when it is taught and practiced on a systematic basis. Although little attention is devoted to this skill in present basal texts, it is beginning to receive more attention. Teachers should encourage the development of this important skill and demand more attention to it in curricular materials.

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Research Resources

MEANING IN MATHEMATICS

Research Findings:

Through meaning, mathematics is seen as a close knit system of understandable ideas, principles and processes. Mathematics instruction is viewed as most effective when students see sense in what they learn. An appropriate balance is needed between meaning and memorization in the learning and retention of mathematics. In the short run, learning derived primarily from rote memorization might develop more efficiently, but that advantage diminishes with time. Drill is not a preferred way of learning mathematics but rather an excellent way to consolidate mathematical learning - provided understanding comes first. It is important to expose students to the kind of practice that involves the retrieval of skills learned several days earlier, rather than practice that follows immediately on the heels of learning. All students should not receive the same amount of practice exercises. It is important to individualize practice as much as possible.

The intimate relationship among time, meaning and retention is essential. Students are more likely to retain mathematical learning over time if meaningful, developmental activities comprise at least half of their mathematics instruction. Teachers should aim for a regular and consistent blend of practice with meaningful instruction.

Questions to Consider:

Do I design the getting started (warm-up, etc.) to include review of pre-requisite skills needed for the lesson of the day, definitions, procedures?

Do I include, in a spiral fashion, review of previous work in each homework assignment?

Do I have students write complete solutions to problems and not just the answers both in classwork and on tests?

Do I have students explain (not just read) their work to the class?

Do I place emphasis as to the meaning of a question?

Do I include summary questions in my lesson plan that will check whether the aim of the lesson was accomplished?

Do I ask summary questions in the lesson to check whether the students have achieved the objectives of the lesson?

Do I phrase questions in the lesson to allow greater opportunities for students to think critically?

Do I use the chalkboard effectively so that procedures, rules, defintions, key ideas, etc. are written for better student notes and future studying?

Do I teach a developmental lesson and provide a model of a problem on the chalkboard before having students practice at their seats?

Are the practice classwork problems varied and graded in difficulty?

Do I have students practice only 2 or 3 problems at their seats, as I walk around helping individuals, have the problems written at the board and explained by students, then repeat this procedure with another 2 or 3 problems, etc.?

VERBALIZATION

Research Findings:

Putting words to one's thinking often brings that thinking to the conscious level

- Students who were asked to verbalize what strategies they had used to solve a set of problems were more successful on a subsequent set of related problems than students who had not been asked to put their strategies into words.

Questions to Consider:

Do I have students write on the board, come to the board and verbalize the strategies for solutions to the homework and to the class problems?

Do I have students respond in full sentence answers so that they are reinforcing not only what is done to solve a problem but also when the method is to be applied?

Do I have students write on the board, come to the board and explain strategies used in solving test questions?

QUESTIONING

Research Findings:

Effective questioning skills have been linked with pupil's achievement in mathematics. Effective teachers of mathematics ask more questions than do ineffective teachers. They ask more recall questions, and they also ask many more higher-level, process-type questions, calling for explanations rather than only recall. Moreover, effective teachers ask more follow-up questions after a correct answer is given, to probe pupils understanding and help them build concepts. Effective teachers encourage pupils to ask questions. This practice leads students to continually evaluate their own understanding. More effective teachers provide more wait-time for answers to a question.

Putting the values on quick correct responses does not help to establish an environment in which the process of mathematical thinking is encouraged. Questioning is an important part of the teacher's ability to establish a classroom atmosphere conducive to the development of mathematical thinking.

Questions to Consider:

Do I prepare questions in advance?

Do I ask more process questions (calling for explanations) than recall questions?

For example:
Do teachers phrase questions to include such terms as "Explain..., Describe..., How..., Illustrate..., Evaluate..., Why..., Compare..., Contrast..., etc.

Do I have students respond in complete sentences?

Do I ask follow-up questions after a correct response is given?

For example:
What evidence do you have for thinking that?
What method did you use for that problem?
How did you get that answer?
What questions does this information leave unanswered?

How do you think your method would work on this next problem? Will this always be true?
How can we get the same answer by using a different reasoning strategy?
Are there other methods of solving this problem?
How could you use what you've learned from solving Problem A to attack Problem B?

Do I encourage students to ask questions? (e.g., what if,...how does this compare with, etc.).

Do I encourage students to question other students or comment on other students' responses?

Do I allow adequate wait-time following a question?

Do I follow-up student responses with the question "why?"

Do I word questions in such a way to involve all of the students in the class?

Do I ask other students to repeat, comment, or elaborate on student reponses (if necessary)?

Do I ask open-ended questions? (e.g. give me an example...)

Do I use questions to get feedback on students' learning?

Do I commend students for correct responses?

Does the student feel free to risk offering their thinking, even if they make mistakes?

Do I emphasize the need for students to explain the "how" and "why" of a problem's solution?

Do I ask students to explain their approaches and results?

Do I avoid asking questions that call for one-word results?

Do I ask for the answer only to a problem and not the process needed to obtain the answer?

Do I give or ask for answers only to homework or test questions?

Are students asked to judge the reasonableness of their thinking and to defend their solutions?

Do I ask students to listen to each other so that they are trained to judge or comment on each other's responses?

Do I ask students to make predictions, formulate generalizations, consider how ideas can be expanded or shifted, look for alternative approaches, and search for new areas to investigate?

ACTIVE LEARNING

Research Findings:

Effective teachers try to keep most of their students from slipping into passive learning. The higher the number of student-initiated questions and comments, the higher was student achievement in mathematics.

Questions to Consider:

Do I ask questions that call for a more sustained student response?

Do I have students play an active part in the explanations to class problems, homework problems and test problems i.e., by having students come to the board to explain the process used and react to other student responses?

Do I return student questions to the other members of the class for reactions, corrections, additions, etc?

Do the students feel comfortable to ask questions?

Do I plan the "Do Now" exercise so that it is a review of the pre-requisite skills, definitions and procedures needed for the new lesson of the day.?

Do I plan a "Do Now" exercise so that it is attainable by most students and can be discussed shortly after the period begins?

Do I use the developmental approach in teaching by having students discover rules and procedures and make generalizatons?

MODELING

Research Findings:

There is a clear correlation between the frequency of mathematics teachers' use of conditional reasoning in their speech (e.g., "if-then" and "whether-then" statements), and their speech (e.g., "if-then" and "whether-then" statements), and their students understanding of logical statements. Students need ample opportunities to use logic and language.

The use of counter-examples of mathematical statements, in conjunction with (positive) example of those statements, resulted in higher achievement than did the use of examples alone.

Questions to Consider:

Do I emphasize to the students the need for knowing both what methods to use in solving a problem as well as when to use the particular method?

Do I have students either use the "if-then" form when responding or give complete sentence responses?

Do I require students to write justifications to their homework solutions and class problems in the "if-then" format?

Do I not interpret students' responses but provide a counter-example for student reaction?

Do I encourage student reactions to student responses?

Do I ask students to give counter-examples to prove or disprove their conjectures?

ERROR ANALYSIS

Research Findings:

One important aspect of a teacher's job is to determine how a student goes about getting the answer. When errors are made, careful examination of them may reveal patterns that are quite logical to a student as well as the teacher. Careful analysis of errors can be done through observation and interviews with individual students.

Questions to Consider:

Do I use feedback wisely so the session serves as a useful diagnostic tool?

Do I ask a student how he or she arrived at an answer?

Do I analyze both the right and wrong answers?

Do I require all work shown with a problem in order to understand a student's thought process and to analyze students' thinking?

Do I identify the types of errors made? (e.g. wrong operation, factual error, incorrect procedure, etc.)

Do I emphasize that listening to each other will help determine errors in thinking?

Do I provide the student with an uninterrupted opportunity to orally discuss his or her explanation of the example?

Do I remain non-judgemental while a student explains his or her solution to a problem?

Are students asked to locate the error point of students' oral or written work?

Do I assign one or two problems to review a single objective?

Are students' assignments made up of a variety of problems dealing with several objectives to support student retention?

WHEN REMEDIATION IS RECOMMENDED:

Do I encourage the student to use materials such as dimes, squared materials or other environmental materials to redo the example?

Do I allow students to use a variety of methods to decompose numbers in order to solve the problem?

Do I provide practice in the recognition of digits in different places in a number?

Do I provide practice in adding tens, hundreds or thousands to a number?

Do I provide varied games to assist in the mastery of concepts?

Do I have other students explain the error made in a problem and explain the correct method of solution?

HOMEWORK

Research Findings:

More effective teachers assign homework and check homework regularly. Less effective teaching results if answers only are given to the previous day's assignment. Students should spend more than half of their time on tasks they can carry out with high success.

Questions to Consider:

Do I construct homework assignments in such a way as to provide each student with high success activities by including review questions?

Do I plan homework in a cyclical fashion so that students frequently review past topics?

Do I plan the homework assignments so that it includes both verbal and non-verbal problems?

Do I lightly stress the topic taught on a given day in the homework so that the teacher may determine the effectiveness of that day's lesson and continue to include this topic cyclically in future homework assignments?

Do I have all homework problems written on the board by students as other students are working on "Do Now" exercises?

Do I use the overhead projector for review of homework by having students prepare their solutions on transparencies?

Do the homework problems written on the board include the actual question as well as all steps in the problem's solution?

Do the students come to the board to explain their solutions and reasoning used in the problem, as well as to respond to students' questions?

Are the students in the class asked to compare their solutions with those at the board, asked to find the points of error, asked to explain the necessary changes to the student who is at the board doing the explanaton of his/her solution?

Do I view the student's organization of solutions to emphasize logical thought and to help other students reduce the amount of careless mistakes?

Are alternative solutions to problems investigated?

While reviewing homework, do I ask summary questions in order to assist students in comparing and contrasting procedures?

Do I create an atmosphere so that students are not fearful in making mistakes or not being able to do the complete solution?

Is the checking of homework an integral part of my instructional program?

TIME ON TASK

Research Findings:

More effective teachers use about three fourths of each period for demonstration and discussion, and only about one-fourth of each period for individual seatwork.

Questions to Consider:

Do I prepare solutions for homework and class problems in advance so that a comparison of the students work may be made more quickly?

Do I have students write and explain reasons to justify their solutions to problems?

ACTIVITY-BASED LEARNING

Research Findings:

Successful programs are those in which student involvement is evident in the process of learning mathematics. This involvement is more than intellectual: the student is actively involved in doing or in seeing something done. Activity-based instruction is concerned with a teaching strategy that has activities that vary widely, form actual real-world experiences to working within groups to accomplish a task. The activity may serve to motivate, to introduce, to provide reinforcement or practice. The activity is necessary to help students apply a mathematical idea to the real world. Programs which support activity-based instruction use varying instructional styles, multimedia approaches, variable-sized groupings and learning (interest) centers.

Questions to Consider:

Do I introduce real-life experiences related to the interests and activities of his/her students?

Do I have students describe their experiences with concrete materials?

Do I incorporate mathematical games and puzzles into mathematics lessons?

Do I encourage the use of the classroom store, newspaper, sports activities and other applications of mathematics in their classrooms?

Does the classroom have a flexible atmosphere, in which students understand that they may be frequently moved in order to facilitate their improvement in mathematics?

MANIPULATIVES

Research Findings:

Lessons involving manipulative materials will produce greater mathematical achievement than will lessons in which manipulative materials are not used if the manipulative materials are used well. Manipulative materials should be used frequently in a total mathematics program in a way consistent with the goals of that program. The manipulatives should be used in conjunction with other aids, including diagrams, textbooks, films, and similar materials. The simplest possible manipulative material (i.e. egg cartons or counters) should be used in direct relationship to the mathematical attribute being studied. Manipulative materials should aid in orgainizing content. Through their use, the student should be aided in recording results symbolically.

Questions to Consider:

Do I have a clear understanding of both the activity and its instructional purpose?

Do I actively participate during these sessions using manipulatives by assisting students who require it?

Do I integrate the use of manipulatives frequently in my classroom?

Do I demonstrate a new activity with the whole class?

MOTIVATION

Research Findings:

Students frequently discern unspoken teacher expectations, and that those unspoken expectations can mold both teacher and student behavior in the classroom. If the expectations are inappropriate, then the resulting behaviors will quite often stand in the way of student motivation.

The teacher's potential for affecting student motivation, positively or negatively, is awesome. While, in the end, motivation must come from within each student, it can only come when the student feels the excitement of learning, experiences his or her efforts as appreciated, gets some clarity on goals, makes some connection between the work done in the mathematics class and those goals, and feels the confidence and freedom to risk attaining them.

Questions to Consider:

Do I use colored chalk to highlight important concepts and motivate students towards learning key ideas?

Do I make use of historical items in mathematics to motivate students interest?

Do I express interest and enthusiasm in mathematics?

Do I allow students sufficient time to formulate guesses and discuss these in class?

Do I provide mathematical puzzles to stimulate student interest in mathematics, to promote discovery and to develop modes of thinking?

Do I appropriately praise students for work in order to motivate students?

REMEDIATION

Research Findings:

Remedial instruction must be different from previous instruction. Multisensory experiences are important for motivational as well as developmental reasons. Students must be gradually guided form the concrete, intuitive understanding of mathematical principles toward the ability to represent their understanding verbally and symbolically.

Questions to Consider:

Do I make mathematics meaningful and relevant to each student by tieing it to students' experiences, hobbies and interests?

Do I use mathematical recreations to stimulate student learning in mathematics at the same time providing practice on mathematical topics?

Do I emphasize study skills, notetaking and test taking skills to the student in need of remediation?

Do I use information from tests to determine which students are in need of remediation?

TEACHER MONITORING

Research Findings:

When monitoring of students by teachers is high, students are more effectively kept engaged in their assigned tasks, which leads to higher achievement.

Questions to Consider:

Do I view students boardwork and notebooks checking for organization, of their work, and providing individual assistance to students when necessary via questioning?

Do I circulate about the room as students are working on the "Do Now" exercises?

Do I analyze students responses to determine student difficulties and thought processes?

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William J. Bennett, Secretary
1986

Mathematical

Concept

Development

Resources

Dot Method for Renaming Fractions

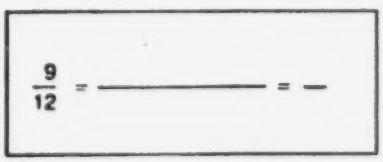
By **Rudy B. Beede**

If renaming fractions is difficult for your students, try using the dot method. Renaming fractions by the dot method has been used successfully with students from grade 3 through grade 12. The method requires no prior knowledge of multiplication facts, common factors, or greatest common factors. It can be used to introduce renaming at the manipulative level in a meaningful way prior to moving to a more abstract level where prime factorization will be involved.

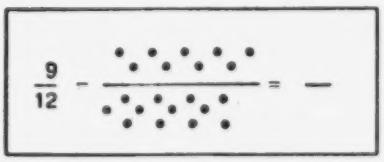
Manipulative Level

A laminated "renaming board" (fig. 1), 40 chips, and a felt-tip marking pen are needed. As an example, let's try to reduce $\frac{9}{12}$.

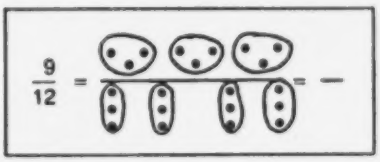
- Write the fraction to be renamed on the renaming board.



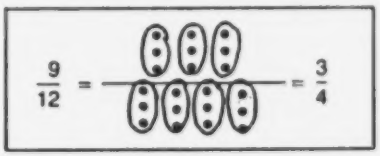
- Place the number of chips corresponding to the numerator and denominator on the renaming board.



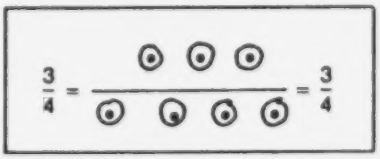
- Group the chips in the largest possible *equal* groups. *Group size must be the same for both the numerator and the denominator.*
- Circle each group with the marking pen.



- Record the number of groups in the numerator and denominator.



- Check for simplest terms by taking the answer from the previous step and representing it with chips, then regrouping (if possible).



Representative Level

To reduce the fraction $\frac{8}{12}$, make a dot picture.

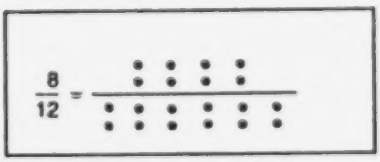
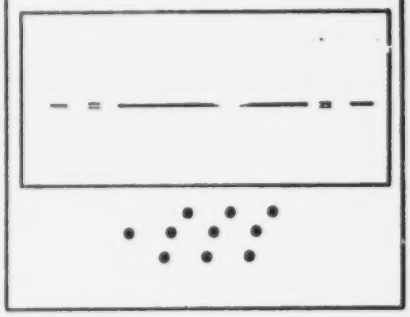


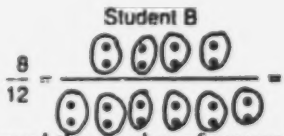
Fig. 1 A "renaming board" with chips



- For the next three steps, possible answers for two different students will be shown. Circle the fewest possible equal groups. *The group size must be the same for both the numerator and the denominator.*

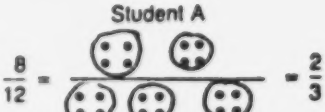
Student A

Student B

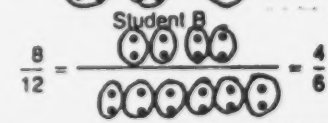


- Record the number of groups in the numerator and the denominator.

Student A



Student B



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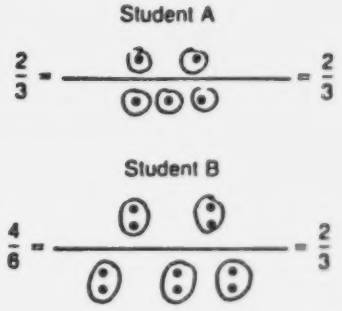
181 151

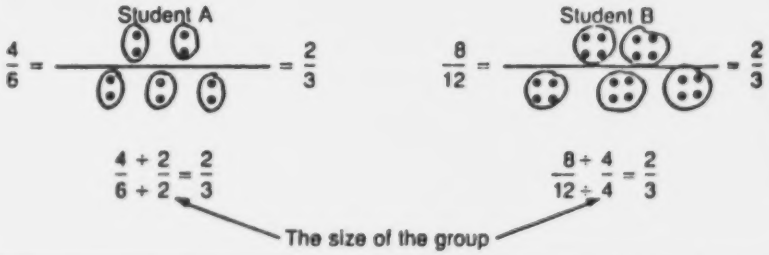
Check for simplest terms by taking the answer from the previous step and making a new dot picture. Regroup if possible

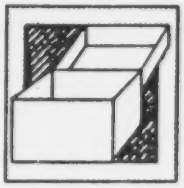


Transition to the Abstract Level

After the student becomes familiar with the dot method, introduce a transitional step as shown in figure 2. After practicing with the dot method, the student should be able to master renaming fractions at the abstract level. ❤

Fig. 2 A transitional step





From the File

Computation

TEAM MATH GAME

This game can easily be adapted for review or drill of any mathematical skill.

Materials: Ten to twenty numbered strips of paper with six problems on each. Also, the teacher will need an answer sheet with all the numbered strips.

Procedure: Organize the class into teams of four or five students (fewer students than the number of problems on the strips). Encourage them to plan how the six problems will be solved—possibly each person doing one problem and then the early finishers working on the remaining ones. Each team picks a spokesperson, who will read the agreed-on answers only to the teacher. The teacher then hands out the same numbered strip to each team and calls for answers after an appropriate amount of time. As soon as an answer is read incorrectly, the teacher moves on to another team. If the team gets all the answers correct, the round is over, and new strips are passed out. The teacher picks the teams in a different order each round. The team that wins the most rounds is the overall champion.

From the file of Christine E. Gaudino, York Suburban Middle School, York, PA 17402

—Readers are encouraged to send in two copies of their classroom-tested ideas for "From the File" to the managing editor for review —

Developing Numeration Concepts and Skills

By **Marilyn Harrison** *and* **Bruce Harrison**

A key Piagetian concept highlighted by Flavell (1963, 368) is that intellectual development is marked by a gradual transformation of overt actions into mental operations. The carrying out of actions by pupils with the decreasing support of external objects can be facilitated by having them use physical objects that exemplify given concepts, patterns, or operations; followed by pictorial or semisymbolic representations; and then imagined objects and operations while keeping symbolic records. Subsequently the abstract concepts emerge in a form that can not only be used meaningfully but also be reinterpreted, as required, in terms of previous levels of representation, rather than exist purely at a symbolic level.

The activities included in this article have been used successfully in leading students from concrete experiences through pictorial and symbolic representations of important numeration concepts and patterns, including place value, multiples, and rounding.

Race to 100

This is an excellent activity for helping children develop clear ideas about the place-value patterns in our base-ten numeration system.

Concept or skill

Understanding of place value and relationships between tens and ones in numbers up to 100

Materials

One die for each pair of students

Base-ten blocks or bean sticks

One place-value board for each student

Method

This game can be played by pairs of students, who take turns rolling a die for units (or beans), exchanging groups of ten for longs (or bean sticks) until the winner reaches 100 (ten longs, a flat, or a bean raft). Teachers can learn a great deal about the students' understanding of place value by observing them as they play Race to 100 and its inverse, Race to 0 (from 100).

What's My Number? (Grade 3)

This activity offers opportunities to apply and extend understanding from Race to 100.

Concept or skill

Identification of the number of thousands, hundreds, tens, and ones in a number

Materials

One place-value board for each child

cards (one for each pair of children)

One set of numerals (0–19) for each pair of children

Place-value material (e.g., units or beans, longs or bean sticks, flats or bean rafts, blocks or bean stacks—up to a value of 999 for each child, with thousands available)

Paper and pencil

Method

Have the children work in pairs. Each child has a place-value board that, in the beginning can be folded back so that only the hundreds, tens, and ones are visible.

1. Give each pair of children one hundreds-tens-ones card and a set of numerals (0–9). Have them shuffle the numerals and place one in each blank (e.g., 1 hundred 3 tens 2 ones). See figure 1.

2. Each child arranges on a place-value board the number of flats or bean rafts, longs or bean sticks, and units or beans indicated by the card.

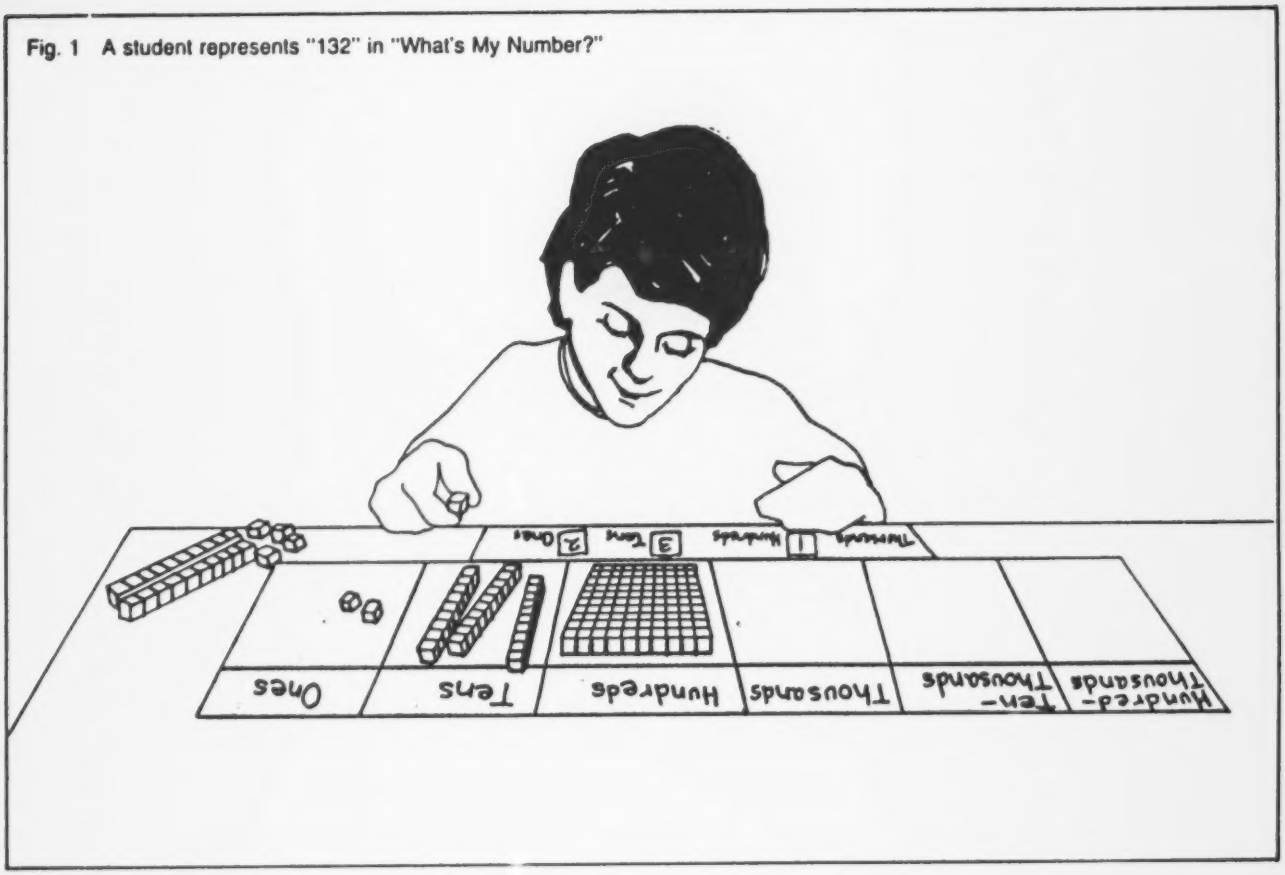
3. Each child writes the number represented, turning the paper over so the partner cannot see what has been written.

4. When both children have finished, they read the number they have written (e.g., "one hundred thirty two") to see if they agree. If not, they discuss the problem.

5. When they have completed several turns without difficulty, add the

Marilyn Harrison is a mathematics consultant with the Calgary Public School Board, Calgary, AB T3J 4M2. She was the project teacher at the Highwood Math Lab from 1976 to 1982. Bruce Harrison is a professor of mathematics education at the University of Calgary, Calgary, AB T2N 1N4. His main interest is in applying insights from cognitive learning theory to mathematics classrooms.

Fig. 1 A student represents "132" in "What's My Number?"



numerals 10–19. (The 10–19 cards can, at first, be limited to the ones place.)

6. Since two-digit numerals can now appear in the place-value blanks, each player must make the appropriate exchanges (e.g., ten beans for a bean stick or ten units for a long).

7. After all the exchanges have been made, each student writes the number as before, then reads it to the partner.

8. When they have completed several cards without difficulty, the rules are changed. Now, after placing the numerals on the card, they write the number they think it will make—before they make it. Then using the place-value material, they make the exchanges and compare. Agreed? If not, they discuss it.

This procedure can lead naturally into thousands: exchanging ten flats for a block or putting several elastics around ten bean rafts to make a "bean stack." Ask the children to fold out their place-value boards to show the thousands place.

Variations

Grade 1: Use only tens-ones cards

_____ tens _____ ones

and the numerals 0–9.

Grade 2: Begin with tens-ones cards and progress to hundreds-tens-ones cards. Use the numerals 10–19 only in the ones place.

Grades 4–6: Begin with hundreds-tens-ones cards but progress quickly to thousands-hundreds-tens-ones cards and then ten thousands-thousands-hundreds-tens-ones cards. With these latter cards, you will probably quickly run out of place-value materials. Use a modified rule: Place the numbers on the cards and have each partner write down the number. Students build the number with place-value materials only if they do not agree.

Students may be interested in visualizing a million by outlining a meter cube with metersticks and placing base-ten blocks (not bean-stick material) inside. Another variation is to use bases other than ten.

High Jinx

This is a good follow-up activity for two to four players that was devised in the Highwood Bilingual School Math Lab in Calgary, Alberta.

Concept or skill

Reading and ordering numerals

Materials

Cards for the numerals 0–9 (at least three sets for each pair of players)

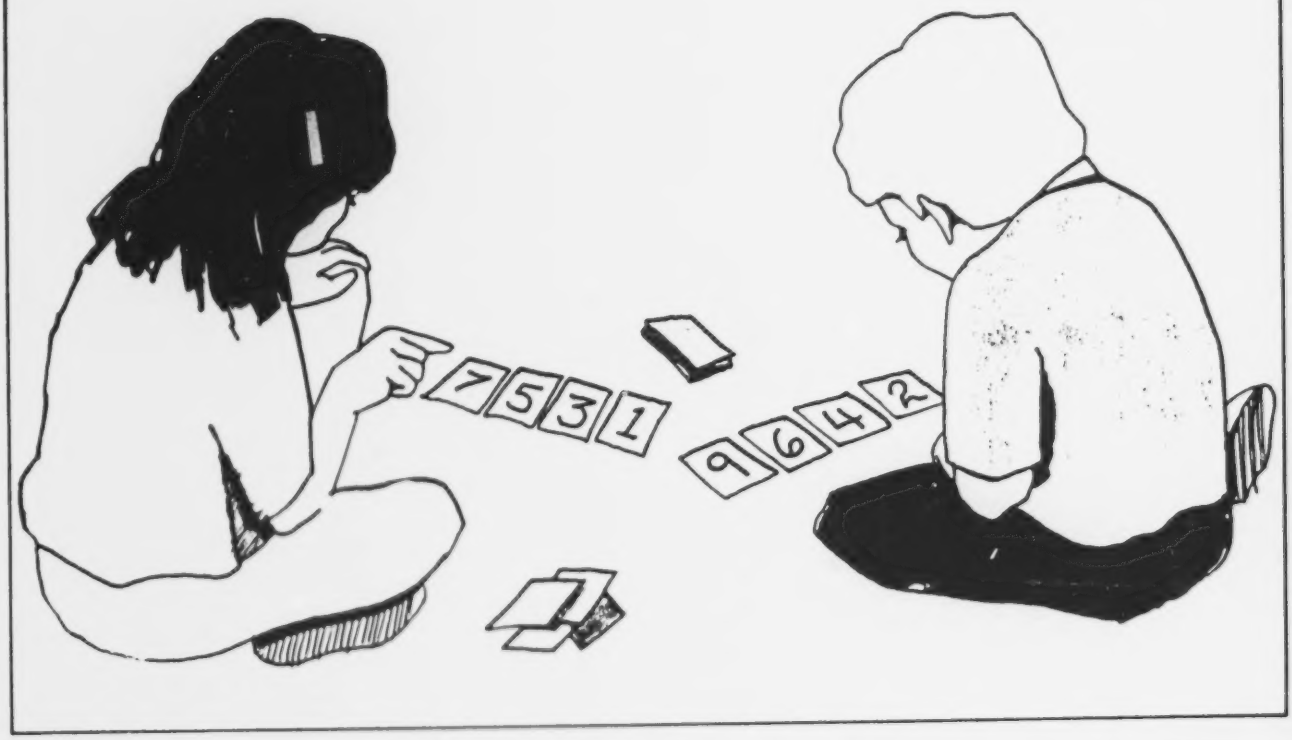
Method

1. Deal 2, 3, 4, 5, 6, or 10 cards to each player for grade 1, 2, 3, 4, 5, or 6, respectively.

2. Make the largest number you can with the cards and lay them down. See figure 2.

3. Read your numbers aloud in turn.

Fig. 2 Students should make the largest number possible with their cards in "High Jinx."



4. The player with the largest number declares, "I have the largest number."

5. The player with the largest number wins all the cards of the other players.

Winner: The player with the most cards when all the cards have been dealt

Variations

a) Least number wins

b) Grades 3–6: Each student also receives a card with a decimal point printed on it. Students are asked to make the greatest number possible, including tenths, hundredths, or thousandths.

Multiples (Grades 2, 3)

This numeration activity, which involves the use of centicubes and rulers, has been found effective for building ideas about place-value and multiplication patterns.

Concept or skill

Counting by 2s, 5s, 10s, 25s, and 100s

Materials

Thirty centicubes for each child
One centimeter ruler for each child
One hundred chart for each child
One centimeter tape measure for each group of five to ten children

Method (for counting by 5s)

1. Each child is asked to put groups of five centicubes together, with each group a different color.

2. Each child places the centicubes along a centimeter ruler, counting by 5s (checking against the numbers on the ruler, as necessary).

3. Students "pool" and link their 5s to count along a tape measure beyond 100.

4. Color in "5," "10," "15,". . . . on the hundred chart. See figure 3.

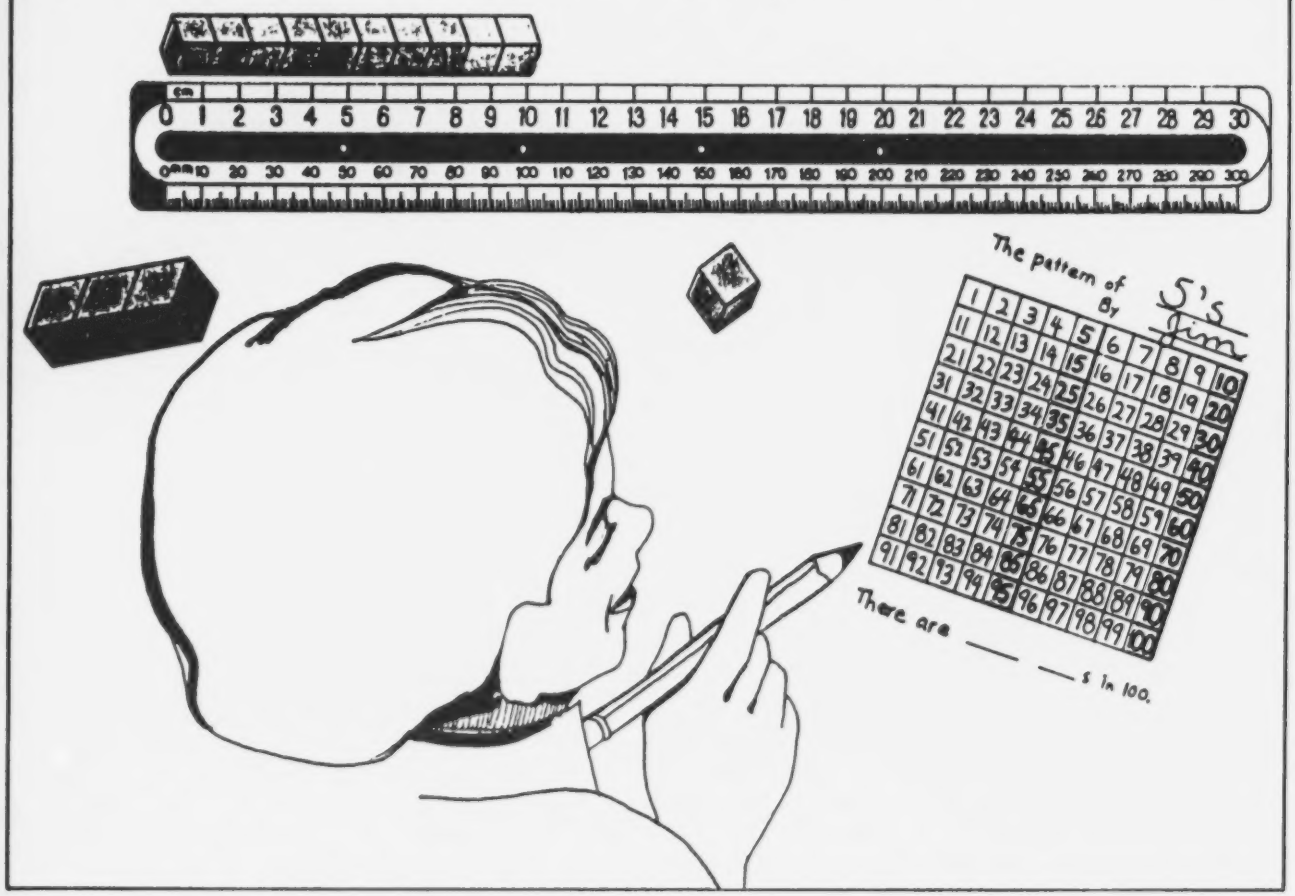
5. Count objects by 5s from containers (e.g., junk boxes; Workjob II Boxes—see Baratta-Lorton [1978]). This activity allows the teacher to observe whether the student can use this skill to count objects. For example, when a teacher observed two students in a heated discussion, it was discovered that one student was, in fact, counting "5" and placing out five objects, "10," and taking ten *more* objects, and so on.

The counting-by-5s procedure can readily be adapted for counting by 2s, 10s, 25s, and 100s.

Rounding (Grade 4)

An area in which many children have traditionally experienced difficulties is rounding, perhaps because it has invariably been presented only at a symbolic level. This activity with structured manipulatives has helped clear up much of the "mystery" about

Fig. 3 Students should color in 5, 10, 15, . . . on the hundred chart in "Multiples."



rounding for children who have found the process difficult.

Concept or skill

Rounding whole numbers to the nearest ten, hundred, or thousand

Materials

Base-ten blocks or bean sticks

Place-value board for each child

Five sets of 0–9 numeral cards, each a different color, for each pair of children

For each pair of children, a set of three cards that read as follows:

"Round to the nearest TEN."

"Round to the nearest HUNDRED."

"Round to the nearest THOUSAND."

Method

Begin with just three sets of 0–9 cards and the direction card "Round to the nearest TEN."

1. Place each set of 0–9 cards facedown.

2. Player A turns over the top card in each of the three piles and builds that number on his or her place-value board using base-ten blocks.

3. Player B writes down what he or she thinks the number will be when "rounded to the nearest TEN."

4. Player A rounds to the nearest ten using this procedure: If the number in the ones place is 4 or less, remove all the units; if the number in the ones place is 5 or more, remove the units (ones) and add another long (ten). One reasonably effective rationalization for the arbitrary rule "Five or more you round up" is this: If you were traveling in an airplane and had gone exactly halfway to your destination and then discovered you had just enough fuel to turn back or go ahead, what would you do?

5. Players check to see if they agree. If not, they discuss. They also record their work:

Number	*Rounded to the nearest ten*
473	470
587	590

When appropriate, have students use five sets of 0–9 cards and these directions: "Round to the nearest TEN." ". . . HUNDRED." ". . . THOUSAND."

Variation 1 (grade 5)

Use six sets of 0–9 cards and add another instruction card: "Round to the nearest TEN THOUSAND."

Variation 2 (grade 6)

Use nine sets of 0–9 cards and add

(Continued on page 60)

(Continued from page 21)

instruction cards for rounding to the nearest HUNDRED THOUSAND, MILLION, TEN MILLION, or HUNDRED MILLION.

This activity can be followed with Catalog Shopping, in which, for example, each student has $20 to spend and must estimate, by rounding, which five items can be bought.

Conclusion

To help students develop mathematical abstractions, it is important to allow them to have experience with physical objects, to discuss that experience, to use and recognize pictures that represent that experience, and, finally, to use symbols to record the experience. "A mathematics textbook for children, however carefully prepared, can be concerned *only* with the last two items of the sequence, pictures and symbols. No book for young children can start where they need to start, namely with experience and spoken language" (Liebeck 1984, 16). It is encouraging to know that Skemp, in England, is presently preparing a set of over 400 carefully sequenced mathematics learning activities (forthcoming) for children in their first seven years of school. These materials will help teachers provide experiences with physical objects necessary for developing abstractions and for promoting productive discussions among students and teachers

Change for a quarter.
Cambio de a 25 centavos.

	D	N
1	2	1
2	1	
3	0	

	D	N	P
4	2	0	

	D	N	P
5	1	2	
6	1	1	
7	1	0	

	D	N	P
8	0	4	
9	0	3	
10	0		
11	0		
12	0		

How many ways can you find to make change for $1.^{00}$ without pennies?
¿De cuántas maneras puedes cambiar un dólar sin usar monedas de centavos?

	H	Q	D	N
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				

	H	Q	D	N
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				
26				
27				
28				

	H	Q	D	N
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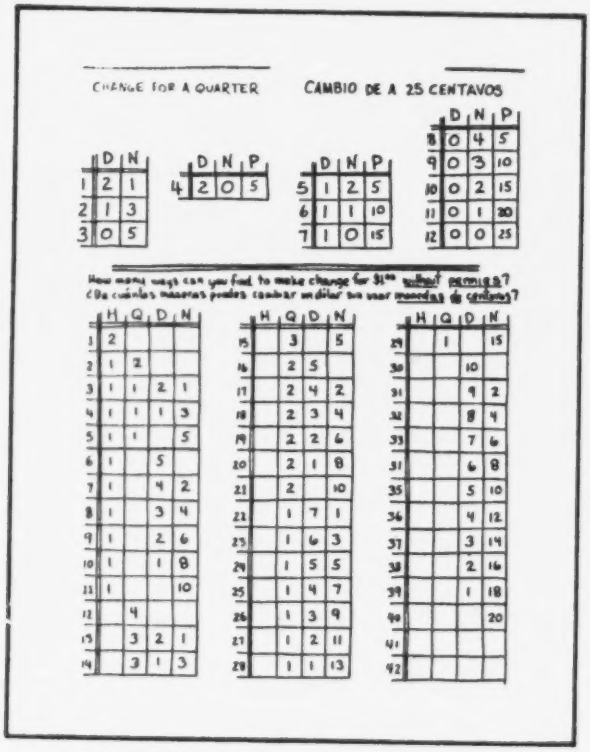

Introducing the Problem

This activity is the first of a series of problems which investigates various coin combinations. The problem above the double line on this page asks about ways to make change for a quarter, and helps children see the importance and usefulness of some kind of a plan or strategy for solving such a problem. While random trial and error might eventually lead to a solution, it can also breed confusion and frustration. The problem above the double line can be used to introduce children to the notion of developing a plan for solving a problem. The question is: "How many ways are there to make change for a quarter, using nickels, dimes and pennies?"

The plan suggested by the charts simply considers four kinds of combinations:

(1) combinations without pennies

(2) combinations with pennies:

(a) using two dimes

(b) using one dime

(c) using no dimes

It turns out that there are just 12 combinations.

52a

The main problem below the double line on this activity page asks for combinations of coins worth exactly $1.00, which do not include any pennies. Children are free to select their own plan to get a solution. Two popular plans have the following beginnings:

H	Q	D	N
2			
1	2		
1	1	2	1
1	1	1	3
1	1	0	5

H	Q	D	N
			20
		1	18
		2	16
		3	14
		4	12

One plan seems to be designed to get combinations using the big coins out of the way first. The other is concerned first with finding all combinations which do not use quarters or halves.

Talking about the Problem

What different plans did children develop? Time should be taken to discuss each one. Some of the discussion might take place in the context of an "educated guessing game". A child shows the first few items in his or her record, and the other children try to guess the plan used.

Consider the following plan:

H	Q	D	N
2			
1	2		
1	1	(2)	1
1	1	1	3
1	1	0	5
1	0	(5)	0
1	0	4	2
1	0	3	4
1	0	2	6
1	0	1	8
1	0	0	10

There is a noteworthy feature of this plan. Once the third combination (one half dollar, one quarter, two dimes and one nickel) has been established, the next two combinations follow directly. The two dimes can be exchanged, one at a time, for two nickels. When five dimes come up in the sixth combination listed, the next five combinations can be easily found by exchanging one dime at a time for two nickels. At this point, all combinations that use a half dollar are exhausted. So, we can go on to combinations that

use quarters — first four quarters; then three quarters, two dimes and one nickel. Then, we can resume the strategy of "turning in" one dime for two nickels.

H	Q	D	N
	4		
	3	2	1
	3	1	3
	3	0	5
	2	5	0
	2	4	2
	2	3	4
	2	2	6
	2	1	8
	2	0	10
	1	7	1
	1	6	3
	etc.		

When we exhaust combinations using quarters, we start on dimes. The combination of ten dimes and no nickels yields the last ten combinations, if the strategy of exchanging one dime for two nickels is continued.

H	Q	D	N
0	0	10	0
		9	2
	etc.		

All together, there are 40 possible combinations of coins that make $1.00, none of which use pennies.

Extensions and Surprise Endings

Now let's look back to the problem above the double line which considers change for a quarter. We could have approached it in another way. There were three combinations that did not use pennies. One nickel was used in the first of these three combinations, three in the second, and five in the third. If we consider each of these combinations in turn and employ a "trading in" strategy for each, there will be nine new combinations. In the first combination, the one nickel can be traded for five pennies. In the second combination, each of the three nickels can be traded in turn for five pennies, to yield three new combinations. The third combination which used five nickels will yield five new combinations.

D	N	P
2	1	0
	0	5

D	N	P
1	3	0
	2	5
	1	10
		15

D	N	P
0	5	0
	4	5
	3	10
	2	15
	1	20
		25

Thus nine new combinations (1 + 3 + 5 = 9) could be derived from the three combinations that did not use pennies, and all together there will be 12 different combinations. This line of reasoning provides a short way to figure the possible combinations for a quarter.

We could have noticed that the original three combinations without pennies have a total of nine nickels.

D	N
2	1
1	3
0	5
	9

Each nickel can be turned in, one at a time, for pennies. Thus there will be nine combinations, and 3 + 9 = 12

Now suppose that we allow pennies to be used in our problem about change for $1.00. How many new combinations can we find? If you start out looking for them one at a time, you will probably break down, bog down, or get a different result each time. There is a much shorter and surer method. All new combinations can be seen as a result of starting with a combination that included nickels, and then trading those nickels, one at a time, for pennies. The 40 combinations without pennies use a total of 252 nickels, so:

40 without pennies
252 with pennies
292 all together

The activity on the reverse side of this page is taken from *Drill and Practice at the Problem Solving Level* (Curriculum Development Associates, 1974)

Store-bought Mathematics

By **Leonard Burkett** *and* **Edie L. Whitfield**

Children often experience difficulty learning mathematics unless it is related to objects or situations that are meaningful to them. Underhill (1981) reports mathematical research (Brownell and Moser 1949; Gray 1965) indicating that memory is greatest for learners when the material to be remembered has meaning for them. Mathematics often has little meaning for children when these relationships are not formed. It may seem remote and removed from real-world events and experiences. Teachers seeking to improve mathematics teaching and learning may wish to consider creating a supermarket math lab in the school or classroom. The supermarket setting can provide an experience-based environment for activities and make mathematics a part of the child's world. This project can be adapted to kindergarten and first-grade activities and can evolve into advanced concepts and problems found in upper elementary school mathematics.

Initiating the project can begin with

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a few sample grocery products arranged in a realistic display on shelves in the classroom. Items such as empty food boxes or empty cans (washed) and a supply of play money can set the stage.

The most basic concepts introduced in numeration are *counting* and *one-to-one correspondence*. Ample opportunity can be afforded for developing these concepts by counting objects on a shelf, counting all the stock, doling out items to a small group, or allowing each member of a small group to select one object. What is happening can be verbalized and recorded on experience charts.

Children can experience notions about the four operations on whole numbers in several ways. They can learn that *addition* is involved when groups of products are placed together to form a larger group (three boxes of oats, four cans of tomatoes, two packages of rice). When different items are used, care should be taken in describing the result. When groups of the same items are put together to form a larger group, three cans of corn and two cans of corn, for instance, the result can be named five cans of corn. Prices of small items can be added to find the total cost (a can of corn for 29¢ plus a box of prunes for 48¢).

They can learn that *subtraction* comes into play when items are removed and they must determine how many are left (seven cans of soda are on the shelf and we remove three). Making change is a higher-level subtraction problem. Two cautions should be noted here: keep the transactions simple at the initial stages, and monitor the process to ensure that the transactions are correct. (A can of peas costs 31¢. If I give the storekeeper 35¢, how much change do I receive?)

They can learn that *multiplication* is helpful for computing the total cost of a group of items of the same price. The grocery store setting is a good opportunity to show that multiplication is a shortened form of addition by doing an example both ways. A can of beans costs 24¢. How much will three cans cost? This can be solved by adding 24¢ + 24¢ + 24¢ or by multiplying 3 times 24¢.

Division can be illustrated by having the students determine unit prices when they are buying by groups. (Six grapefruit cost 78¢. How much does one cost?)

After focusing on simple examples that illustrate only one operation, problems can be formulated that involve more than one operation:

> Three children are purchasing supplies for their camping trip. The storekeeper adds up the list (addition) containing groups of like items (multiplication). The children give him $20, and the storekeeper gives them change (subtraction). Each child pays an equal share (division).

In this situation, the three campers and the storekeeper are identified, and the situation is carried to completion. Other children, as well as the teacher, monitor the process.

Decimals become easier to understand when they are used to solve problems that actually occur in a busy supermarket. Children have been exposed to money through television, shopping trips, and other experiences. Introducing decimals in the context of buying at the supermarket can impart a clear understanding of the need for, and the use of, decimal numbers.

Learning to operate a cash register or a calculator gives students many opportunities to practice basic operations with decimal numbers. (If we buy meat for $2.39 and eggs for $0.98, how much does our purchase cost? If we spend $5.36 for soap and coffee and soap is $0.68, how much is the coffee? If we buy eight boxes of cookies for $1.29 each, how much will we spend? Or, if we pay $6.89 for twelve candy bars, how much does one candy bar cost?)

A natural introduction to percentage occurs when children need to learn to compute sales tax. (We spent $42.50 for groceries. How much is the 5-percent sales tax? We spent $36.58 for groceries and $8.29 for medicine. Medicine is tax-exempt. How much was the total tax for the purchase? What was the total cost, including tax, of the purchase?) Other percentage problems arise when we compare supermarket specials. (Is it cheaper to purchase an $0.89 item with a 10-percent discount or a $0.95 item with a 15-percent discount?)

Fractions are needed in the supermarket lab when children calculate parts of pounds, gallons, or bushels. Such products as oleo and eggs lend themselves to development of the concept of equivalent fractions (4/12 of a dozen eggs is equal to 1/3 dozen).

Ratio and proportion can be used to compare prices and to determine the best buy on differing amounts of food for differing amounts of money. (Which is cheaper, 1/4 pound for $0.30 or 1/2 pound for $0.50? Three oranges for $0.25 or four oranges for $0.30?)

Word problems are often the most difficult for elementary school children. One method for helping children learn to solve word problems is suggested by Crowder and Wheeler (1972): rewriting textbook examples using situations from the student's experiences as a basis. Rewriting problems using the supermarket theme is an ideal method for giving the problems meaning, since children often hear parents and other family members planning food budgets and discussing prices and newspaper ads. Children also profit by creating their own stories about shopping experiences.

A variety of problems can be explored in a store setting. Children can participate in establishing their "store" by bringing in empty cans, bags, wrappers, and boxes. With very little expense and effort, these items can be arranged to simulate shelves in a supermarket. The variety of exercises and the number of concepts that can be developed are limited only by the imagination of the teacher and the students.

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Coordinate Geometry—Art and Mathematics

By **Michael Terc**

Our students cry for self-expression, for a chance to see mathematics in action. Frequently, however, the structure of mathematics does not lend itself to individual style or variation. Problem solving can tend to be dull and monotonous rather than exciting and stimulating.

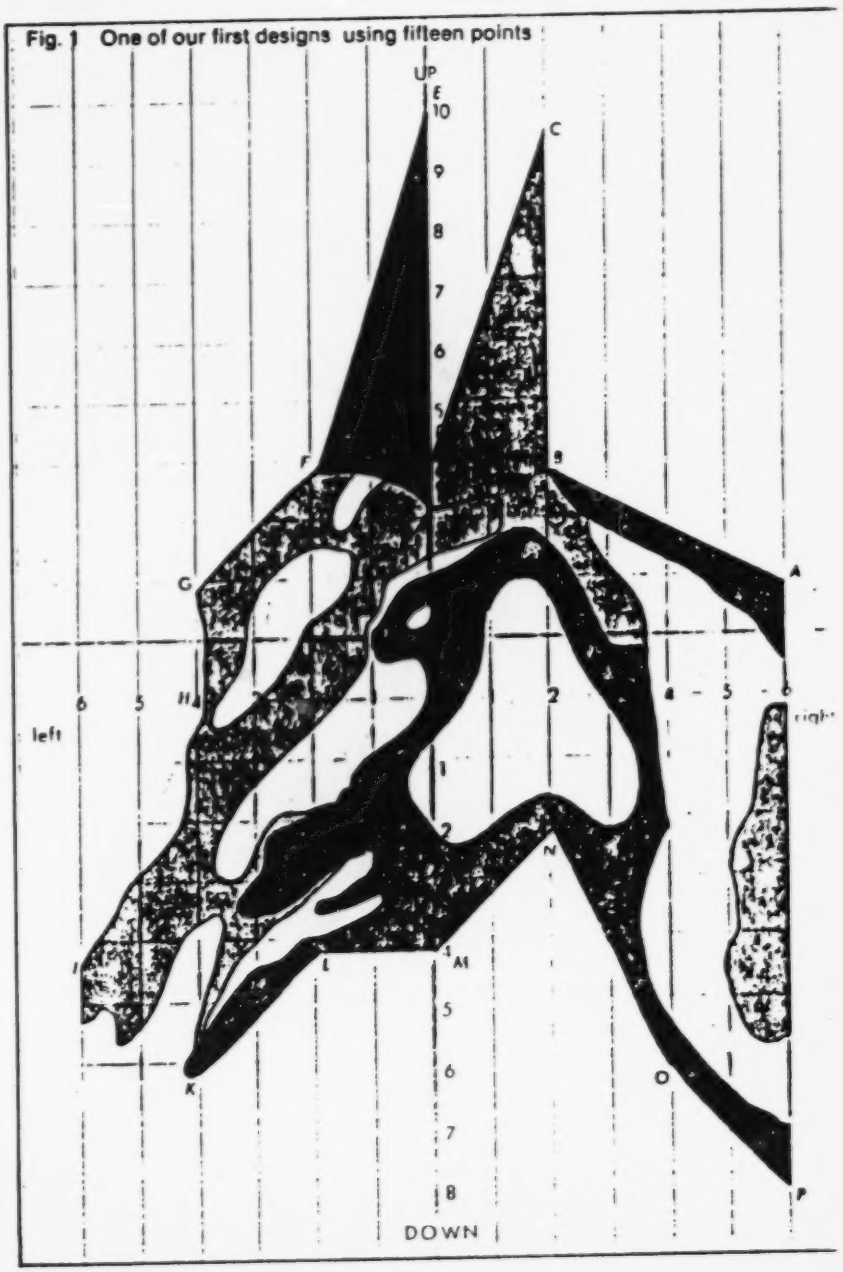
Not all mathematics can be fun and games, but teachers have an obligation to make the subject as enjoyable as possible. If we come upon a topic that nurtures students' involvement and enjoyment, we should capitalize on it.

One such topic that I recently taught has the potential to generate and foster students' involvement and lends itself to real individual effort. I hope this record of my experience and satisfaction will allow me to share my joy and excitement with my colleagues.

The topic is *coordinate geometry*. Basically, coordinate geometry is a combination of algebra and geometry. It is treated in high school mathematics courses, but fundamental concepts can be introduced to students at an earlier age. I hope to dispel the notion that it is inappropriate for the elementary school student.

My first encounter with teaching coordinate geometry was quite en-

Fig. 1 One of our first designs using fifteen points



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lightening. I began by describing the coordinate axes and the method used to plot points on the graph. Many students were unable to appreciate the beauty of Descartes's invention. I decided to arrange my coordinates so that a familiar picture or geometric object would emerge. At first, my patterns were simple. For example, my initial designs were a horse's head (fig. 1), a boat, and a house, each consisting of no more than twenty pairs of coordinates.

Almost everyone enjoyed plotting the points. Pupils were particularly fascinated that mathematics could be used to construct figures. They began to use colored pencils to create distinctive work.

Suddenly, my slow learners' interest increased markedly. For the first time some students were actually self-motivated. I had them create more coordinate pictures with more points. The figures became an astronaut, an Indian, a polygon, and an eagle.

As students' involvement increased, their personal pride began to flourish. Their pictures were really looking quite good! Students with active imaginations began to add their own backgrounds to the figures. A plain whale became Moby Dick with an appropriate seascape, complete with lighthouse and seagulls (fig. 2). Students whose attentions normally were prone to wander when working in an individual setting were transformed into high-powered students concerned with their work and anxious to see their pictures materialize.

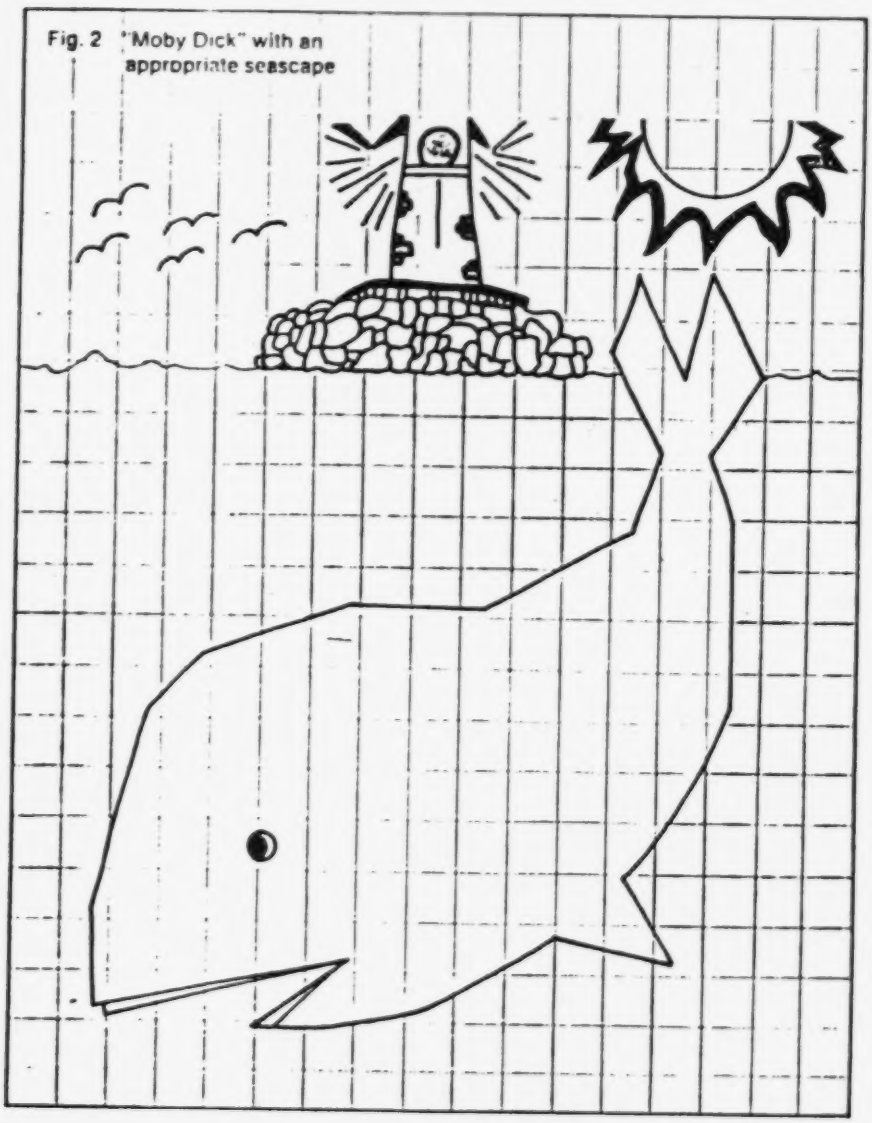

Fig. 2 "Moby Dick" with an appropriate seascape

To capitalize on this enthusiasm, I decided to hold a contest. The rules were simple; students could enter as many figures as they wished. All were encouraged to submit at least one of their best pictures employing their most imaginative skills. Most students enhanced the figures with colored pencils and Magic Markers. Some used glitter, and one student added real feathers. Out of 120 students, ninety entries were received. The classes then selected the top twenty entries, which were then attractively displayed on the bulletin board. The entry of each finalist was assigned a number, and all the students were asked to select their top five choices. The votes were tallied, and the five winners were announced the next day. Winners were awarded extra-credit points on their semester grade averages. As a final touch, the winning pictures were displayed around the room with the outstanding selections from previous years, creating a "mathematics hall of fame."

It was not until last year that I extended the topic on an even grander scale. Our principal challenged the faculty to improve the visual appeal of the building. Teachers were encouraged to consider special displays, bulletin boards, and other aesthetic projects. It occurred to me that we might use our knowledge of coordinate geometry to help draw murals on our classroom walls.

Our first consideration dealt with supplies. Could we obtain brushes, paints, and, more importantly, sturdy and safe scaffolds on which our students could work? The answer was yes! The art department and custodial staff would supply us with whatever materials we would need.

Second, when would the actual work be done? Could it be accomplished without interrupting everyone's schedule? Again the answer was yes! We would map out designs before or after school, and we would paint during the regular school day.

Now all that was needed was a

small group of interested students and agreement about the murals to be drawn. Five student volunteers were selected according to neatness, dependability, and some mathematical or artistic ability. After the selection process was completed, we met and decided on the murals to be drawn. We agreed on a complete color scheme with accompanying diagrams.

Next, the walls were marked lightly in pencil so that they resembled giant graph boards with the lines spaced one inch apart. To insure that the lines were vertical and horizontal, we used a level that we obtained from the wood shop. Scaffolds were used to reach those areas above normal height. The next phase involved plotting the figures using the appropriate coordinates. The points were then connected to produce the desired figure. The third phase was the most exciting. A penciled figure became a beautifully painted whale. Bare walls were transformed into a series of polygons. A once drab corner became a showcase for a majestic eagle (fig. 3). Best of all, these transformations were achieved with mathematics. For the first time, many students realized that mathematics is practical—it can be seen, used, and appreciated as a part of our world!

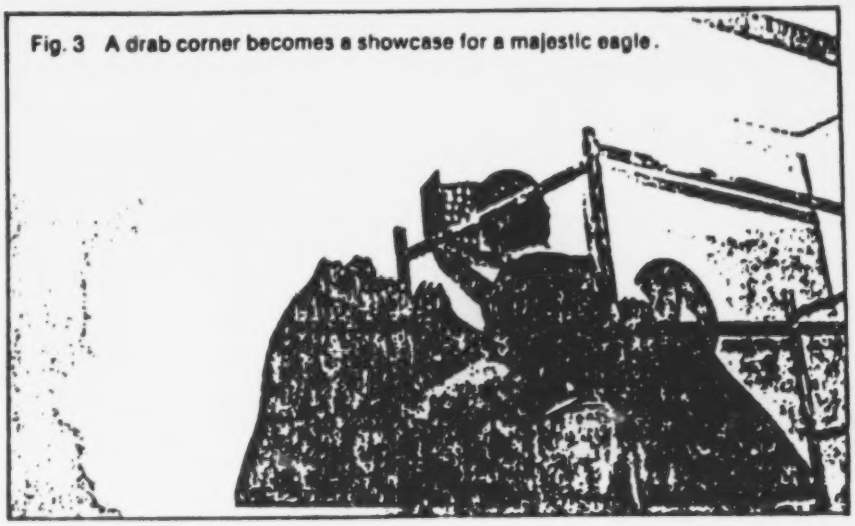

Fig. 3 A drab corner becomes a showcase for a majestic eagle.

This lesson had additional positive side effects. For example, a parent called to say that his son had changed their basement wall into a beautiful sailboat scene. Another youngster used a piece of plywood, some string, nails, and his imagination to create a coordinate figure.

All in all, the coordinate geometry unit has been an exciting and worthwhile adventure through the wonderland of thought that we call mathematics. ●

Multiple Methods of Teaching the Addition and Subtraction of Integers

By **Lisa Chang**

Many of us learned to obtain the correct answers to problems with integers by applying rules. Few of us had a good grasp of the meanings behind these rules. Yet we may be offering the same golden rules, with little meaningful explanation, to our own students.

Computation needs to be related to the concepts of the operations, and both concepts and computational skills should be developed in the context of real-world applications. Instruction on computational skills needs to be meaningful to the learner (Hamrick and McKillip 1978). The rules for the basic operations with integers can be learned after the students have comprehended the concepts involved. This article will outline how this process might be undertaken in teaching the addition and subtraction of integers.

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Addition

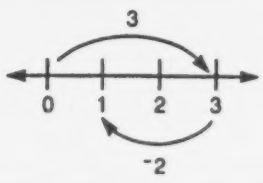
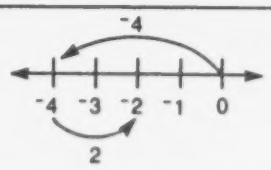
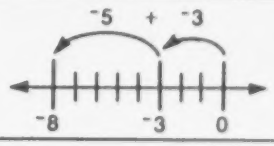
Money technique

Materials. Play money, problem-situation cards

Fig. 1 Money technique for teaching the addition of integers

Integer Problem-Situation Card	Mathematics Sentence	Play-Dollars Solution	Solution
John earned \$8 for mowing the lawn. He then spent \$5 on a birthday gift for his sister.	$8 + {}^{-}5 =$ ———	had \$8 \$5 \$1 \$1 \$1 spent \$3 left	3 (the total money left)
Mary withdrew \$10 from her savings account on Monday. She withdrew \$5 from her savings account on Friday.	${}^{-}10 + {}^{-}5 =$ ———	withdrew \$10 \$5 \$15 $^{-}15$	$^{-}15$ (the total money Mary withdrew from her savings account)
David owed his friend \$3 yesterday. His mother gave him \$5 this morning.	${}^{-}3 + 5 =$ ———	had \$5 \$1 \$1 \$1 \$1 \$1 \$3 \$2 left returned \$3 he owed	2 (the total money left)

Fig. 2 Number line–walking technique for teaching the addition of integers

Integer Problem-Situation Card	Mathematics Sentence	Number Line–Walking Solution	Solution
Start at 0. Move to the right three steps, then move to the left two steps.	$3 + {}^{-}2 =$ _____		1 (one step to the right)
Start at 0. Move to the left four steps, then move to the right two steps.	${}^{-}4 + 2 =$ _____		-2 (two steps to the left)
Start at 0. Move to the left three steps, then continue to move to the left five more steps.	${}^{-}3 + {}^{-}5 =$ _____		-8 (eight steps to the left)

Process. Pick a problem-situation card. Write the mathematics sentence for the card. Use play money to solve the situation on the card. Money earned or money deposited in a savings account represents positive numbers, whereas money spent or money withdrawn from a savings account represents negative numbers. The complete process is illustrated in figure 1.

Number-line–walking technique

Materials. Number line, problem-situation cards

Process. Pick a problem-situation card. Write a mathematics sentence for the problem shown on the card. Solve the problem by walking on the number line. Draw the solution. Finally, record the solution using integers. Point out that steps are counted and that at position zero you have not yet moved and so nothing is to be counted. Moving to the right indicates a positive direction, and moving to the left indicates a negative direction (see fig. 2).

Color-block technique

Materials. Two sets of blocks; poker chips, or some reasonable substitute, in two colors; problem-situation cards

Process. Pick a problem card. Solve the problem concretely using the blocks. Record the concrete solution. Finally, solve the problem symbolically, using integers. One set of blocks represents positive numbers and the other, negative numbers. The sum of one positive block and one negative block equals zero. Hence $[+] + [-] = 0$ (see fig. 3).

Subtraction

Money technique

Materials. Play money, problem-situation cards

Process. Pick a problem-situation card. Write the mathematics sentence for the card. Use play money to solve the problem shown on the card. Money earned or received represents positive numbers, whereas money paid for merchandise or services represents negative numbers. Use the link between addition and subtraction to help solve the problem. Figure 4 illustrates this technique.

Number-line–walking technique

Materials. Number line, problem-situation cards

Process. Follow the number-line process for the addition of integers. Use the links between addition and subtraction to help solve the problems. The examples are illustrated in figure 5.

Color-block technique

Materials. Two sets of blocks, or a substitute, in two colors; problem cards

Process. Follow the color-block process given for the addition of integers. Use the links between addition and subtraction to help solve the problem. The examples illustrating this technique are shown in figure 6.

Fig. 3 Color-block technique for teaching the addition of integers

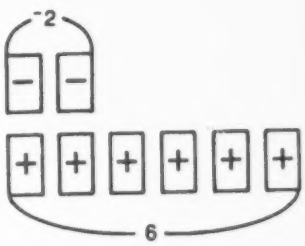
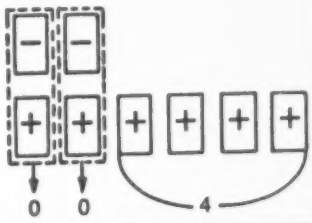
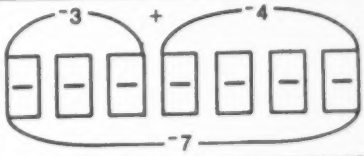
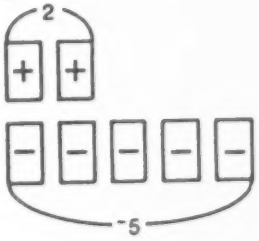
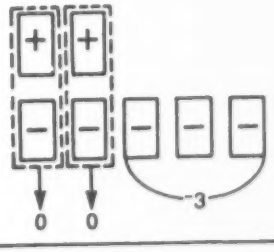
Integer Problem-Situation Card	Mathematics Sentence	Color-Block Solution	Solution
Begin with two negative blocks, then add six positive blocks.	$^-2+6=$ _____	*Step 1*  *Step 2*  	4 (four positive blocks left)
Begin with three negative blocks, then add four more negative blocks.	$^-3+^-4=$ _____	 	-7 (seven negative blocks in all)
Begin with two positive blocks, then add five negative blocks.	$2+^-5=$ _____	*Step 1*  *Step 2*  	-3 (three negative blocks left)

Fig. 4 Money technique for teaching subtraction of integers

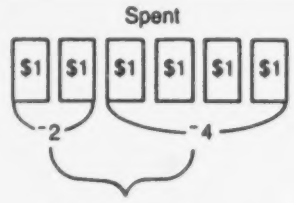
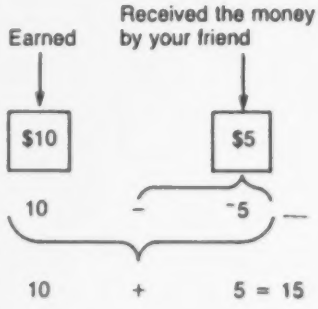
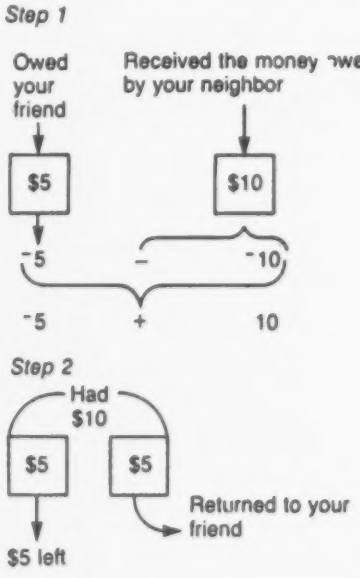
Integer Problem-Situation Card	Mathematics Sentence	Play-Dollars Solution	Solution
Spent $2 for a game ticket, then spent $4 for a birthday gift.	$^{-}2 - 4 = {}^{-}2 + {}^{-}4 =$ _____		-6 (spent $6 in all)
Earned $10 from delivering newspapers. Then received $5 your friend owed you.	$10 - {}^{-}5 = 10 + 5 =$ _____		15 (had $15 in all)
Owed your friend $5, then received $10 from your neighbor who owed you for mowing his lawn.	$^{-}5 - {}^{-}10 = {}^{-}5 + 10 =$ _____		5 ($5 left)

Fig. 5. Number line–walking technique for teaching subtraction of integers

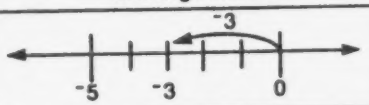
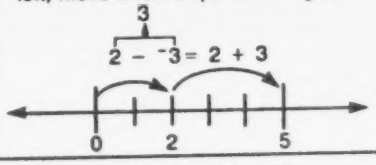
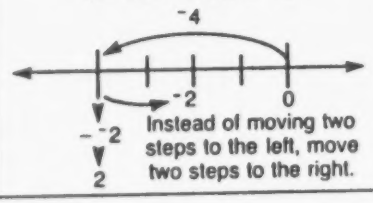
Integer Problem-Situation Card	Mathematics Sentence	Number-line–Walking Solution	Solution
Start at 0. Move three steps to the left.	$0-3=0+{}^{-}3=$ ____		$^{-}3$ (three steps to the left)
Start at 0. Move two steps to the right. Then, *instead of* moving back to the left three steps, move three more steps to the right.	$2-{}^{-}3=2+3=$ ____	Instead of moving three steps to the left, move three steps to the right. 	5 (five steps to the right)
Start at 0. Move four steps to the left. Then, *instead of* going in the same direction two more steps, move to the opposite direction two steps.	$^{-}4-{}^{-}2={}^{-}4+2=$ ____	$^{-}4-{}^{-}2 = {}^{-}4+2 = {}^{-}2$ 	$^{-}2$ (two steps to the left)

Fig. 6 Color-block technique for teaching subtraction of integers

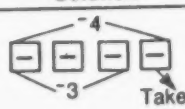
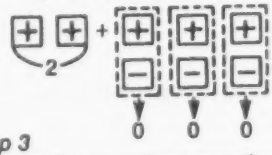
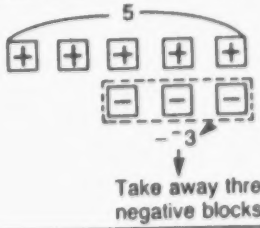
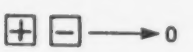
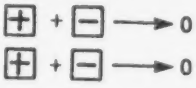
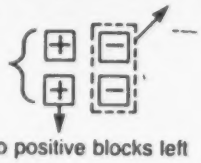
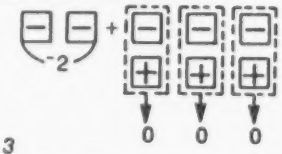
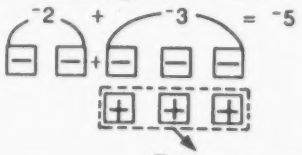
Integer Problem Card	Problem Situation	Color-Block Solution	Solution
$^{-}4 - {}^{-}1 =$ ____	Begin with four negative blocks. Then take away one negative block.		$^{-}4 - {}^{-}1 = {}^{-}3$ (three negative blocks left)
$2 - {}^{-}3 =$ ____	Begin with two positive blocks. Take away three negative blocks from these two positive blocks.	*Step 1* Begin with two positive blocks. *Step 2* To take away three negative blocks ($-{}^{-}3$) add three sets of zeros.  *Step 3* Now take away three negative blocks from two positive blocks.  Take away three negative blocks.	$2 - {}^{-}3 = 2+3 = 5$ (five positive blocks)

Fig. 6 Color-block technique for teaching subtraction of integers —*Continued*

Integer Problem Card	Problem Situation	Color-Block Solution	Solution
$0 - {}^{-}2 =$ _____	Begin with zero (one positive block and one negative block). Take two negative blocks from zero.	*Step 1* Begin with zero.  *Step 2* To take away two negative blocks ($-{}^{-}2$), add another set equaling zero.  *Step 3* Remove two negative blocks from the zeros.  Two positive blocks left	$0 - {}^{-}2 = 2$ (two positive blocks left)
${}^{-}2 - 3 =$ _____	Begin with two negative blocks. Take three positive blocks from two negative blocks.	*Step 1* Begin with two negative blocks. *Step 2* To take away three positive blocks, add three sets of zeros.  *Step 3* Now take away three positive blocks and have five negative blocks left. 	${}^{-}2 - 3 =$ ${}^{-}2 + {}^{-}3 = {}^{-}5$ (five negative blocks)

Conclusion

This article has briefly outlined the paths that might be followed in teaching integers to children who have a wide range of ability. The final "right" answer is not the only consideration. Learning the meaning behind the operations, relating operations with integers to real-world situations, and finding a variety of ways to solve the problems involving integers are also important activities. The techniques outlined here can concretely represent a very abstract notion. Students come to understand the concepts involved, experience success with abstract ideas, and gain confidence in their results by manipulating the concrete objects and participating in these performance-oriented activities.

References

Hamrick, Katherine B., and William D. McKillip. "How Computational Skills Contribute to the Meaningful Learning of Arithmetic." In *Developing Computational Skills*, 1978 Yearbook of the National Council of Teachers of Mathematics, edited by Marilyn N. Suydam, pp. 1–12. Reston, Va.: The Council, 1978.

Professional Standards for Selection and Implementation of Instructional Materials

The following statement is an official NCTM position.
It was developed by the Instructional Issues Advisory Committee and adopted by the Board of Directors.

THE CONTENT, organization, and style of instructional materials, particularly basal textbooks, are among the most important factors influencing outcomes of school mathematics programs. Thus, the process of selecting new textbooks and ancillary materials is a critical step in planning curricula and preparing for instruction. This process includes setting criteria, screening materials, making choices, and implementing the new program. Although teachers and students are vitally concerned with the quality of the instructional materials they must use, the selection and preparation for use of those materials are often constrained by inappropriate selection and implementation processes.

In order to improve curriculum and instruction in elementary and secondary mathematics programs, the National Council of Teachers of Mathematics has adopted the following standards for selecting instructional materials and implementing programs using those materials. The Council urges compliance with these standards by all school districts.

Standard 1: The Review and Selection Committee

The entire process of textbook selection should be led by teachers and supervisors with expertise and responsibility in mathematics education. Except in most unusual circumstances, the recommendations of the committee must be followed.

Membership on the Review and Selection Committee should include, but not necessarily be limited to, (1) classroom teachers representing the grades or courses for which instructional materials are being selected as well as teachers in grades or courses that precede or follow the target courses; (2) mathematicians or mathematics educators with knowledge of content, curriculum, and pedagogical trends covering the full spectrum of K-12 mathematics and beyond; and (3) school administrators who have overall responsibility for supporting and guiding the implementation of new programs.

Selection of committee members must be made in consultation with persons having expertise and responsibility in mathematics education. All persons selected should be full working members. Where the talents of a school or district mathematics staff cannot fulfill the roles described above, appropriate outside consultants should be obtained.

Standard 2: Criteria for Selection of Instructional Materials

Criteria for selection must be defined and made public prior to the review process.

The criteria should identify the desired instructional approach as well as the mathematics content. Criteria for selection should be based on program needs, which can be identified by—

- evaluation of the present program, including analysis of student achievement, attitudes, and learning characteristics and a survey of teacher opinions;
- current research on the teaching and learning of mathematics;
- recommendations of professional organizations;
- curriculum guidelines of local, state or provincial, and national governments.

Difficulties in implementation should be considered only if they seem insurmountable. Any adoption of new instructional materials will create some problems of implementation. However, the committee's goal should be the identification of the best possible program, not the easiest to implement.

Copyright dates and formula-determined reading levels are generally inappropriate as selection criteria. Although fashions in content and pedagogy change, a newer (or older) copyright does not assure higher quality. Because of the large technical vocabulary associated with mathematics, results of standard readability formulas should be used with extreme caution.

Standard 3: The Review of Instructional Materials

The Review and Selection Committee should actively seek materials from a variety of sources and evaluate the materials on the basis of the previously defined criteria.

The process of identifying, collecting, and examining material for potential adoption takes time to be done well. Time for committee members to do their work must be made available by the school system.

To identify appropriate instructional materials, the committee should consult—

- recommendations of teachers and supervisors in the district and in nearby districts having similar characteristics and goals;
- recommendations or authorizations of state or provincial committees where they exist;
- critical reviews in professional journals;
- independent information bases, such as the National Diffusion Network;
- representatives of all publishers of appropriate material.

Instructional materials developed by nonprofit organizations as well as commercial publishers should be considered. Whenever possible, committee representatives should examine materials displayed at a national or regional NCTM meeting. Interviews with members of the author team or a publisher's representative may also be helpful.

Once the materials for review have been collected, each should be evaluated for the—

- content coverage and development of each major curricular strand;
- development of the full range of cognitive abilities, especially problem solving and analysis;
- correlation of the content with local, state or provincial, or national scope-and-sequence guidelines;
- correlation of the content and instructional approach with research on mathematics teaching and learning;
- correlation of the content and instructional approach with the standards of professional mathematics organizations;
- correlation between textual features and characteristics of special students;
- evidence of success based on publishers' field-validation studies, implementation in other districts, critical reviews in professional journals and reports, or local field testing where possible;
- representation of the diversity of national social, political, and economic life.

Since the cost of instructional materials is only a small fraction of any school's budget, the cost of these materials should not be a factor. Potential financial savings do not justify the selection of less desirable materials. Where textbooks are accompanied by ancillary materials, however, it is important to assess the extent to which successful use of the text depends on such extra materials.

Standard 4: Implementation of New Instructional Materials

The selection of new instructional materials must be only part of a careful curriculum-development process that includes appropriate in-service preparation for teachers and supervisors and a formal evaluation of the new program's effectiveness.

In-service preparation should acquaint all teachers with the general program goals and the emphasis for each topic in every course. This emphasis should, where possible, be identified in terms of the appropriate number of instructional days to be devoted to each topic, and the topics should be correlated with specific portions of the new instructional materials. In-service preparation should also provide any necessary additional mathematics background as well as preferred instructional approaches to the mathematics content.

Implementation should include periodic evaluation surveys of teachers' and students' reactions to the materials as well as an analysis of student achievement. Results of these evaluations should be used to evaluate the achievement of the program goals and to plan for ongoing modification of the program. Implementation is typically more successful when such in-service support continues throughout the academic year.

Standard 5: Professional Integrity

The instructional materials selection process must not allow the professional integrity of the process to be compromised.

Any person connected with the process of selecting and implementing instructional materials who has a financial interest in any program being considered must make that interest known publicly at the outset. For the duration of the selection process those involved must not accept gifts of travel, entertainment, or materials from a vendor.

The instructional materials recommended by the Review and Selection Committee must be adopted unless irregularities in the selection process are documented.

(September 1984)

EVALUATION OUTLINE FOR MATHEMATICS TEXTS

This outline has been prepared to assist in the process of selecting mathematics textbooks. Supervisors, teachers, staff developers and students should consider each category when analyzing and comparing texts and ancillary materials. Section IV should be used in conjunction with the District's stated goals for each grade.

I. READABILITITY

A. Clear explanation?
B. Appropriate reading level?
C. Appropriate use of vocabulary and symbols?

II. LESSON DEVELOPMENT

A. Spiral approach used?
B. Is the sequence of lessons logical?
C. Can the sequence of lessons be correlated with that of the course of study?
D. Development of concepts from concrete to pictorial to abstract?
E. Lessons geared for student verbal participation?
F. Examples helpful to understanding?
G. Sufficient written practice follows explanation?
H. Do lessons consider differences in ability?
I. Sufficient practice?
J. Range of levels exist in practice material?
K. Are some answers provided for self-checking?
L. Is there a balance between skill practice and problem solving?
M. Pre-Tests for developmental skill chapters?
N. Lesson review tests?
O. Chapter tests?
P. Unit reviews?

III.

A. Index helpful?
B. Lesson background appropriate?
C. Lesson allows for range of abilities?
D. T.E. assists with error diagnosis?
E. Suggestions for use of teaching aids?
F. Lessons suggest warm-ups for maintenance of previous skills?
G. Homework suggested?
H. Workbook material helpful?
I. Answer key available for pupils?

IV. RATE TREATEMENT OF EACH TOPIC IN RELATION TO GRADE OBJECTIVE (MTE, SCOPE & SEQUENCE, STANFORD DIAGNOSTIC MATH TEST):

A. Problem solving
B. Applicaiton to real life
C. Encourage checking through reasonableness
D. Estimation utilized
E. Computation of whole numbers
F. Computation of fractions, decimals
G. Geometry
H. Metric & customary unit balance

IV. RATE TREATEMENT OF EACH TOPIC IN RELATION TO GRADE OBJECTIVE (MTE, SCOPE & SEQUENCE, STANFORD DIAGNOSTIC MATH TEST): (cont'd)

I. Reading, interpreting & marking graphs, tables, charts
J. Probability used for predicting events
K. Calculators utilized
L. Introduction to Basic; Computer Literacy

V. A. Text attractive? Durable?
B. Are pictures related to the instruction?
C. Uncluttered presentation?
D. Headings clear & understandable?

*This outline is base upon a monograph prepared by the Virginia Council of Teachers of Mathematics to assist in the state's textbook adoption process. The monograph may be borrowed from the Mathematics Office or ordered for $2.00 from the Virginia Council of Teachers of Mathematics, 2143 Colta Mack Court, Reston, Virginia 22091

EVALUATION OF MATHEMATICS TEXTS

Text:

Grade:

Publisher:

Please check answer:

1) Does it correlate with the new "Scope and Sequence?" Yes_____ No_____

2) Are the explanations clear? Yes_____ No_____

3) Does the text present problems that are thought-provoking and require reasoning skills (e.g. not enough information, unnecessary information, require critical thinking, such as determining the proper mathematical operation).

Yes_____ No_____

4) Are there an adequate number of practice problems? Yes_____ No_____

5) Are there enrichment activities? Yes_____ No_____

6) Are there remediation exercises? Yes_____ No_____

7) Is the reading level - on grade _____?

above grade _____

below grade _____

8) Is the sequence of lessons logical? Yes_____ No_____

9) Would you recommend this text? Yes_____ No_____

10) Comments:

_____ _____

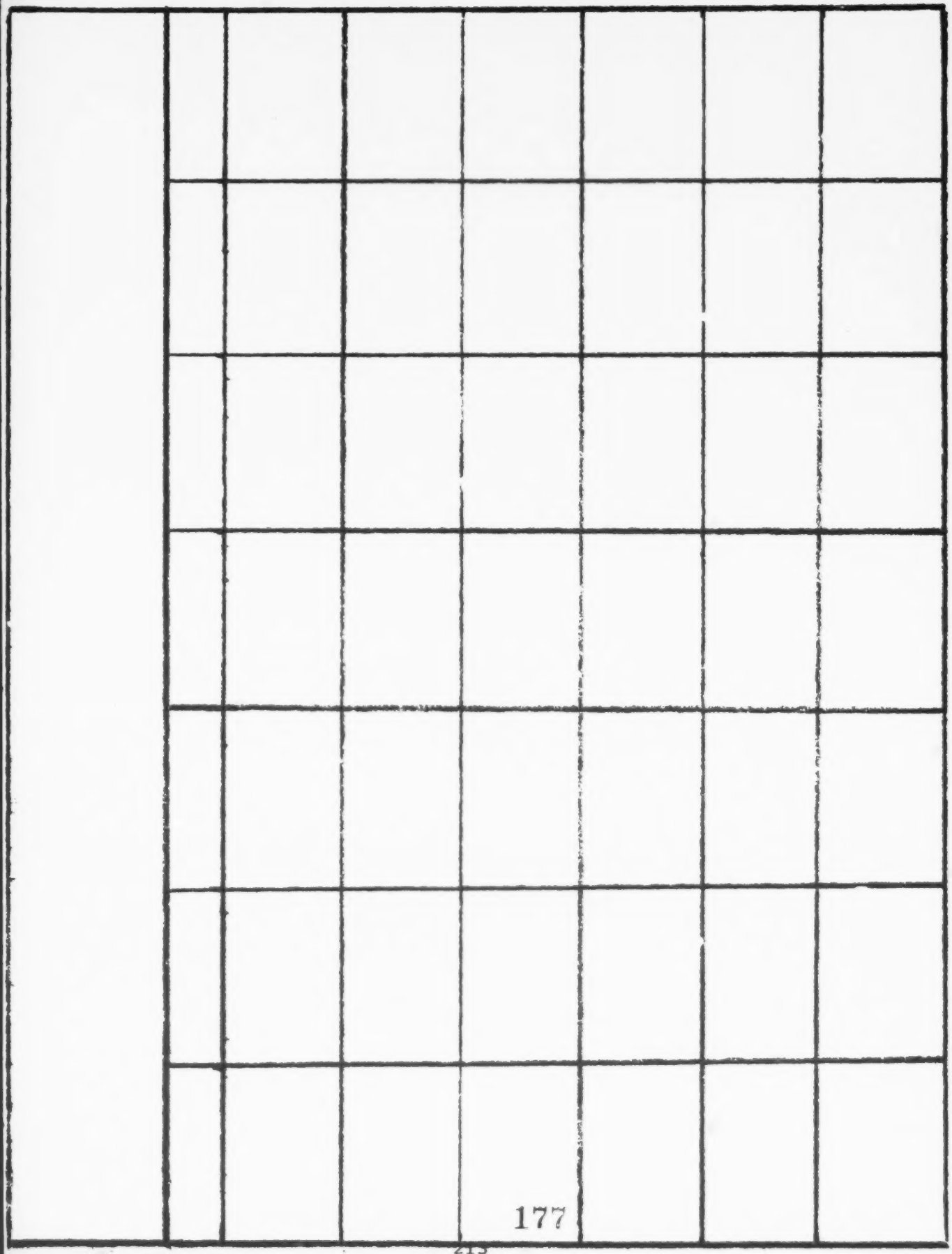
Staff Member School

D#113

Sample

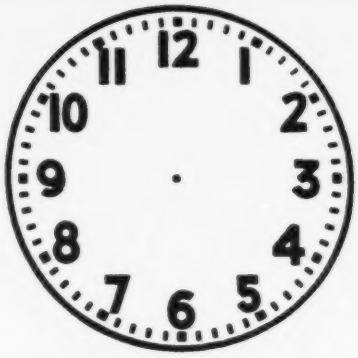
Mathematics

Masters



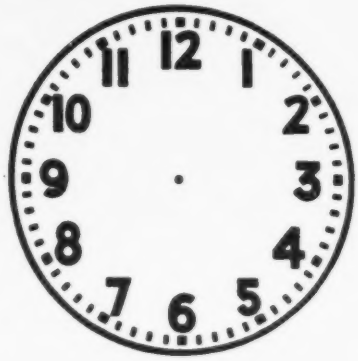
A_____

B_____



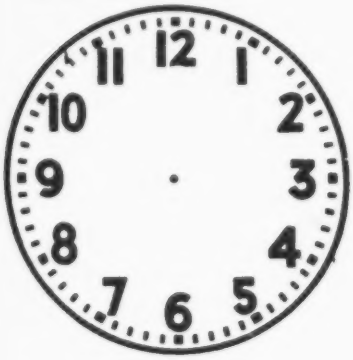
C_____

D_____



E_____

F_____



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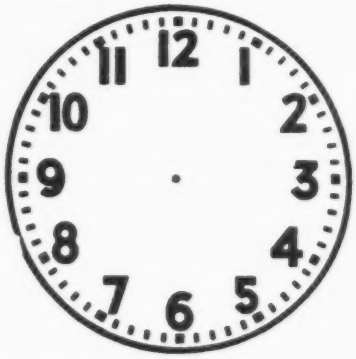
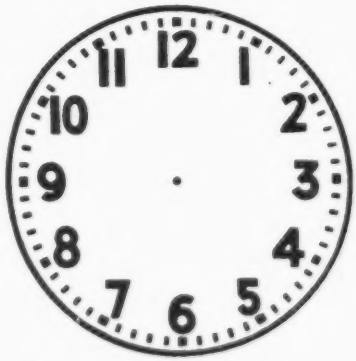
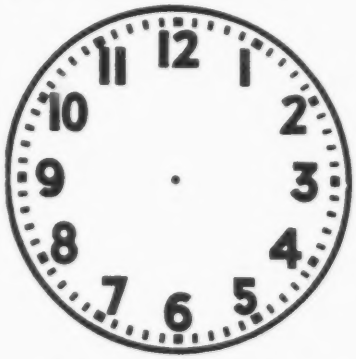


H_____

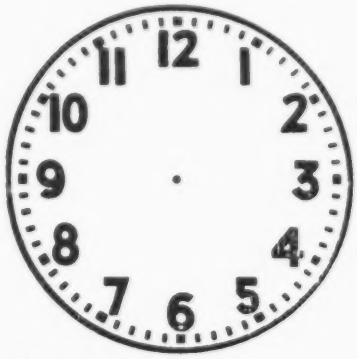


I_____

J



K



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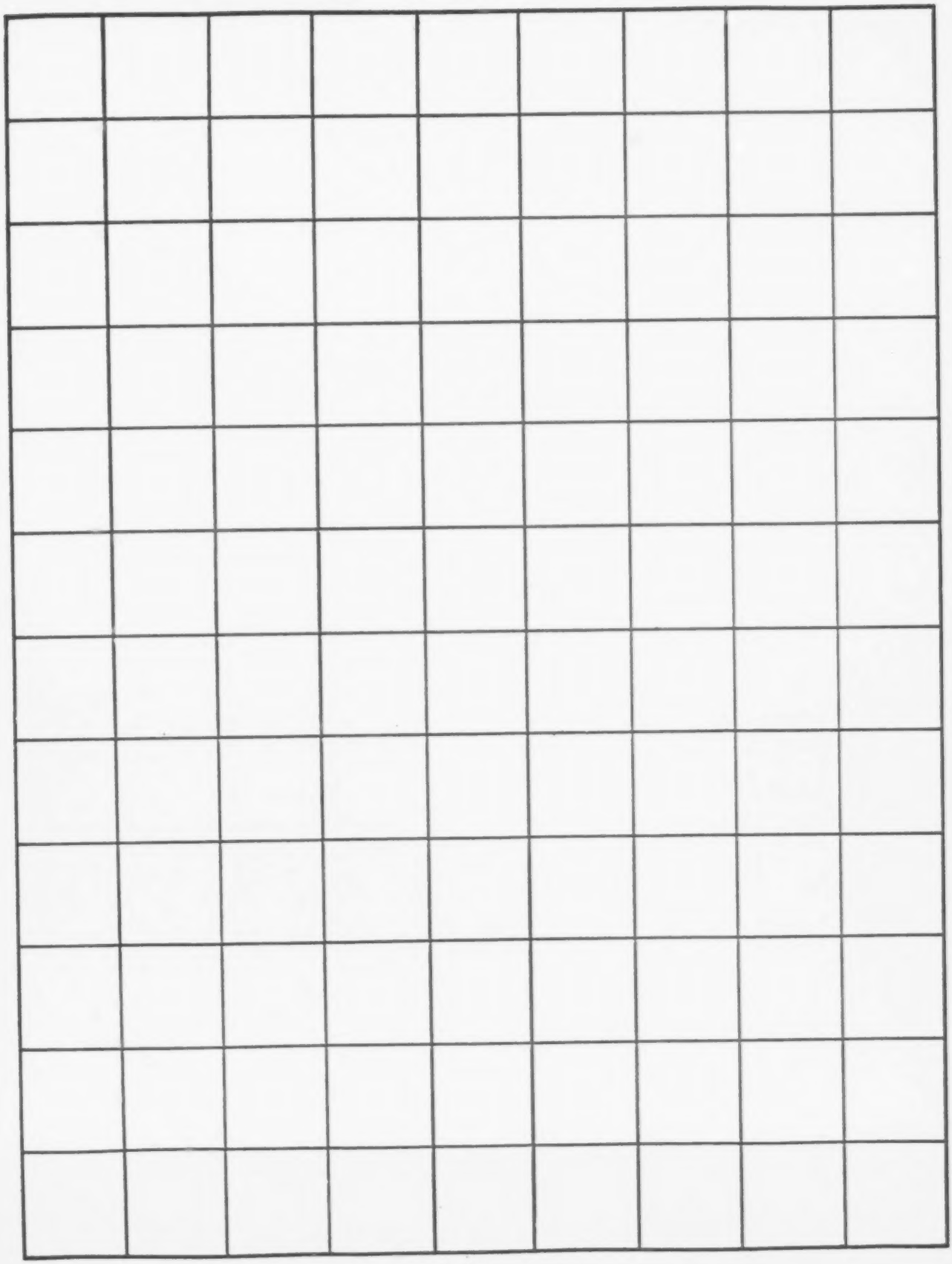

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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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11	12	13	14	15	16	17	18	19	20
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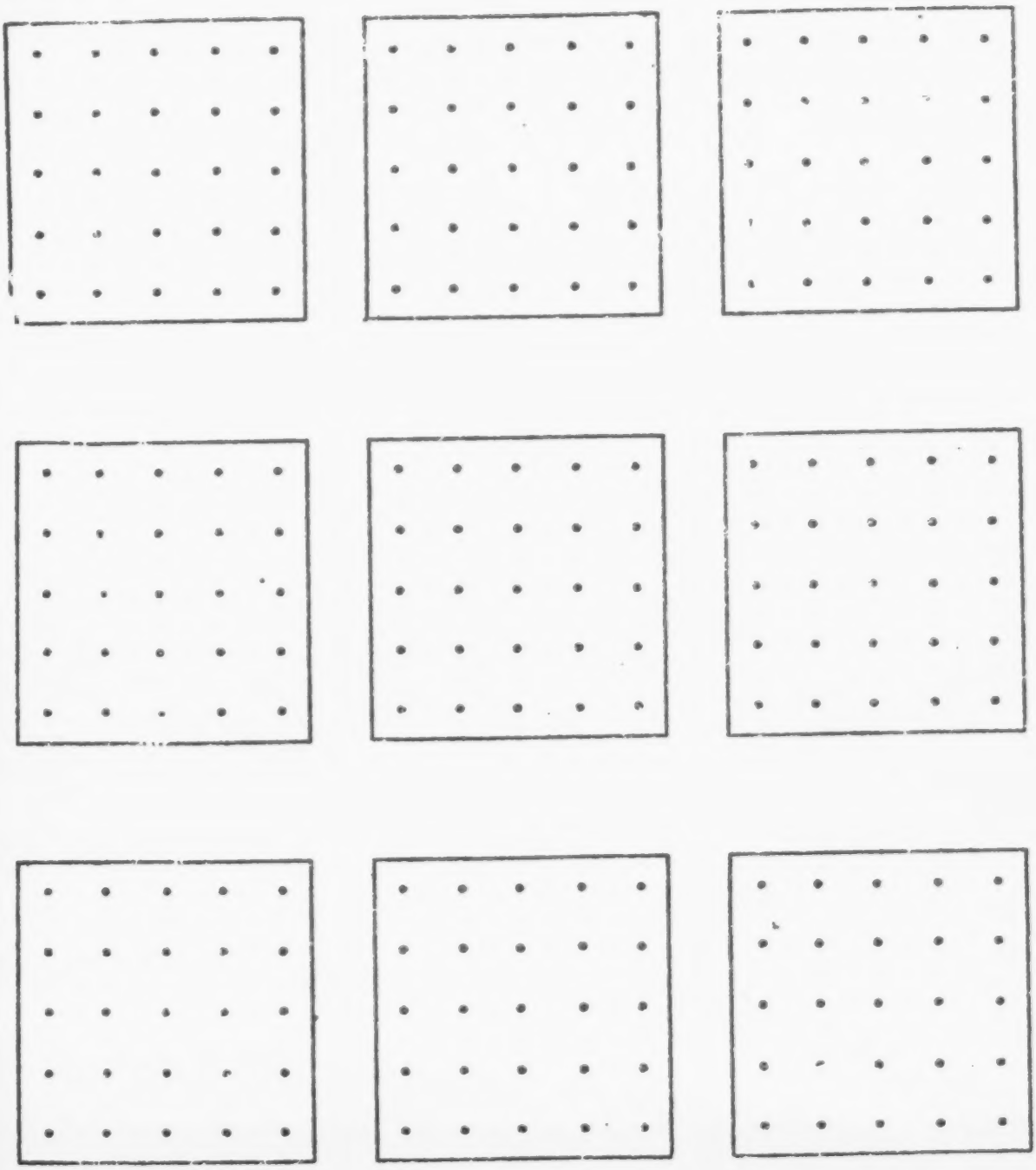
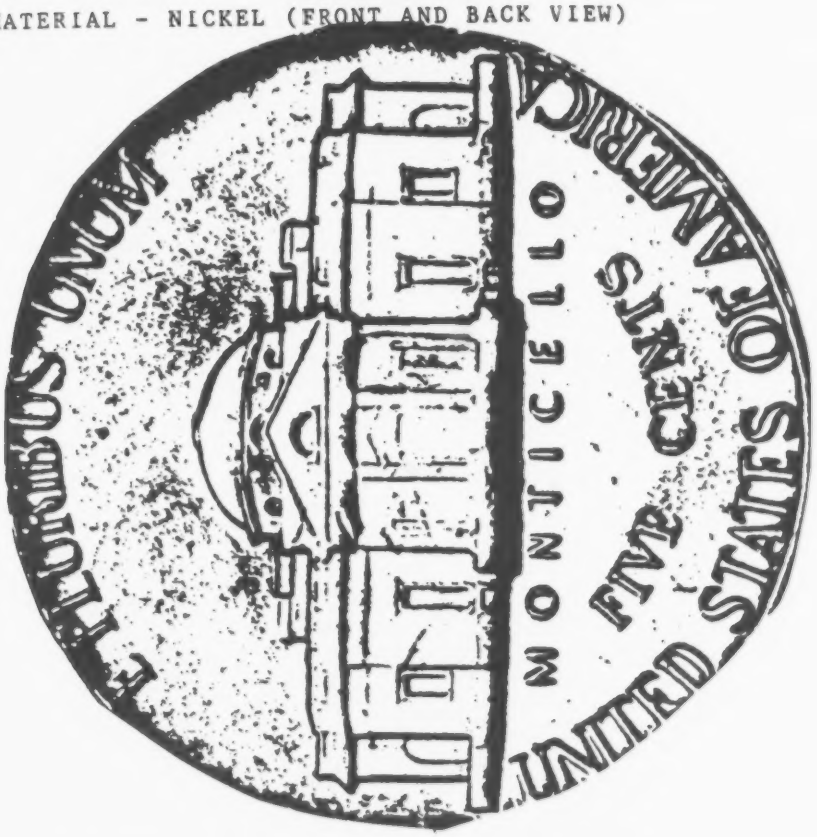
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Geoboard Grids



RESOURCE MATERIAL - PENNY (FRONT AND BACK VIEW)

RESOURCE MATERIAL - NICKEL (FRONT AND BACK VIEW)



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RESOURCE MATERIAL - DIME (FRONT AND BACK VIEW)

RESOURCE MATERIAL - QUARTER (FRONT VIEW)

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RESOURCE MATERIAL - HALF DOLLAR (FRONT VIEW)

RESOURCE MATERIAL - HALF DOLLAR (BACK VIEW)

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